

Spinorially twisted Spin structures, III: CR structures

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Abstract

We develop a spinorial description of CR structures of arbitrary codimension. More precisely, we characterize almost CR structures of arbitrary codimension on (Riemannian) manifolds by the existence of a $\text{Spin}^{c,r}$ structure carrying a partially pure spinor field. We study various integrability conditions of the almost CR structure in our spinorial setup, including the classical integrability of a CR structure as well as those implied by Killing-type conditions on the partially pure spinor field. In the codimension one case, we develop a spinorial description of strictly pseudoconvex CR manifolds, metric contact manifolds and Sasakian manifolds. Finally, we study hypersurfaces of Kähler manifolds via partially pure Spin^c spinors.

1 Introduction

Spinors have played an important role in both physics and mathematics ever since they were discovered by É. Cartan in 1913. We refer the reader to Hitchin's seminal paper [13], Friedrich's textbook [10], as well as to [22, 15] for the more recent development of Seiberg-Witten theory and its notorious results on 4-manifold geometry and topology.

The starting point of this paper was our interest in characterizing and studying CR structures (of arbitrary codimension) by means of twisted Spin structures and spinors. Our main motivation has been the relation of almost complex structures with “classical pure spinors” (and Spin^c structures). Cartan defined *pure spinors* [3, 4, 5] in order to characterize (almost) complex structures and, almost one hundred years later, they are still being used in related geometrical problems [2]. Furthermore, these spinor fields have been related to the notion of calibrations on a Spin manifold by Harvey and Lawson [11, 6], since distinguished differential forms are naturally associated to a spinor field and, in particular, give rise to special differential forms on immersed hypersurfaces. Pure spinors are also present in the Penrose formalism in General Relativity as they are implicit in Penrose's notion of “flag planes” [18, 19, 20].

The notion of abstract CR structures in odd dimensions generalizes that of complex structure in even dimensions. This notion aims to describe intrinsically the property of being a hypersurface of a complex space form. This is done by distinguishing a distribution whose sections play the role of the holomorphic vector fields tangent to the hypersurface. There exists also the notion of almost CR structure of arbitrary codimension, in which a fixed codimension subbundle of the tangent bundle carries a complex structure. It has been proved that every codimension one, strictly pseudoconvex

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CR manifold has a canonical Spin^c structure [21]. Naturally, this led us to ask if it is possible to characterize almost CR structures of arbitrary codimension (and a choice of compatible metric) by means of a twisted Spin structure carrying a special spinor field.

We developed the algebraic background of twisted partially pure spinors in [12] which we recall briefly in the second section. Let us recall, in particular, the definition of twisted Spin group

$$\text{Spin}^{c,r}(n) = \frac{\text{Spin}(n) \times \text{Spin}^c(r)}{\{\pm(1,1)\}},$$

which will be the structure group for the twisted Spin structures (cf. Definition 3.1), and whose representations contain the partially pure spinors. Note that r will eventually be the codimension of an almost CR structure. Such twisted Spin structures involve not only the principal bundle of orthonormal frames, but also two auxiliary principal bundles. The need for such structures stems from the fact that there are manifolds which are neither Spin nor Spin^c . Subsection 3.1 is devoted to showing that there are triples of principal bundles admitting $\text{Spin}^{c,r}$ structures.

The existence of a partially pure spinor field ϕ on a Riemannian $\text{Spin}^{c,r}$ manifold M^n implies the splitting of the tangent bundle TM into two orthogonal distributions V^ϕ and $(V^\phi)^\perp$, where the former is endowed with an automorphism J^ϕ satisfying $(J^\phi)^2 = -\text{Id}_{V^\phi}$, i.e. M has an almost CR hermitian structure. In fact, the converse is also true (cf. Theorem 4.1). Furthermore, we characterize the integrability condition of a CR structure (with metric) by an equation involving covariant derivatives of the partially pure spinor (cf. Theorem 4.2). We proceed to study other natural “integrability conditions” of the partially pure spinor field, such as being parallel in the V^ϕ directions (cf. Theorem 4.3), or being Killing in the $(V^\phi)^\perp$ directions (cf. Theorem 4.5), etc. We present a family of homogeneous spaces as examples for the different theorems.

As mentioned before, the relevant group for codimension one almost CR structures is $\text{Spin}^{c,1}(n) = \text{Spin}^c(n)$. Thus, we prove that partially pure spinors appear naturally and implicitly in extrinsic Spin^c geometry: consider a Kähler manifold endowed with a Spin^c structure carrying a parallel spinor ψ . It is known that the restriction ϕ of the parallel spinor ψ to a real oriented hypersurface M satisfies

$$\nabla_X^M \phi = -\frac{1}{2} \Pi(X) \bullet \phi,$$

where Π denotes the second fundamental form of M , ∇^M is the Spin^c covariant derivative on M and “ \bullet ” the Clifford multiplication on M [14, 16]. Moreover, the spinor ϕ is partially pure and integrable (see Theorem 5.2).

The paper is organized as follows. In Section 2, we recall the background material for the definition of partially pure spinors, and describe the isotropy representation of a family of homogeneous spaces (partial flag manifolds) that will be used throughout the paper. In Section 3, we define $\text{Spin}^{c,r}$ structures, study their existence, define twisted Dirac and Laplacian operators, prove some curvature identities and a Schrödinger-Lichnerowicz-type formula, and derive some Bochner-type results. In Section 4, we give the spinorial characterization of (almost) CR hermitian structures, and examine various integrability conditions and their geometrical consequences. In Section 5, we return to the codimension one case and examine in our spinorial context (strictly) pseudoconvex CR manifolds, metric contact manifolds and Sasakian manifolds, and explore extrinsic geometry questions including immersion theorems (Theorems 5.2 and 5.3).

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2 Preliminaries

In this section, we briefly recall basic facts about Clifford algebras, the Spin group and the standard Spin representation [10]. We also define the twisted Spin groups and representations, the antisymmetric 2-forms and endomorphisms associated to a twisted spinor, recall the definition of twisted partially pure spinor, and describe the isotropy representations of certain homogeneous spaces that will furnish examples later on.

2.1 Clifford algebras, the Spin group and representation

Let Cl_n denote the Clifford algebra generated by the orthonormal vectors $e_1, e_2, \dots, e_n \in \mathbb{R}^n$ subject to the relations

$$e_j e_k + e_k e_j = -2 \langle e_j, e_k \rangle,$$

where \langle, \rangle denotes the standard inner product in \mathbb{R}^n . Let

$$\mathbb{C}l_n = Cl_n \otimes_{\mathbb{R}} \mathbb{C}$$

denote the complexification of Cl_n . The Clifford algebras are isomorphic to matrix algebras

$$\mathbb{C}l_n \cong \begin{cases} \text{End}(\mathbb{C}^{2^k}), & \text{if } n = 2k, \\ \text{End}(\mathbb{C}^{2^k}) \oplus \text{End}(\mathbb{C}^{2^k}), & \text{if } n = 2k + 1. \end{cases}$$

The map

$$\kappa : \mathbb{C}l_n \longrightarrow \text{End}(\Delta_n)$$

is defined to be either the above mentioned isomorphism if n is even, or the isomorphism followed by the projection onto the first summand if n is odd. An expression for κ can be given explicitly using the matrices

$$\text{Id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

In terms of the generators e_1, \dots, e_n , κ is given by

$$\begin{aligned} e_1 &\mapsto \text{Id} \otimes \text{Id} \otimes \dots \otimes \text{Id} \otimes \text{Id} \otimes g_1, \\ e_2 &\mapsto \text{Id} \otimes \text{Id} \otimes \dots \otimes \text{Id} \otimes \text{Id} \otimes g_2, \\ e_3 &\mapsto \text{Id} \otimes \text{Id} \otimes \dots \otimes \text{Id} \otimes g_1 \otimes T, \\ e_4 &\mapsto \text{Id} \otimes \text{Id} \otimes \dots \otimes \text{Id} \otimes g_2 \otimes T, \\ &\vdots \\ e_{2k-1} &\mapsto g_1 \otimes T \otimes \dots \otimes T \otimes T \otimes T, \\ e_{2k} &\mapsto g_2 \otimes T \otimes \dots \otimes T \otimes T \otimes T, \end{aligned}$$

and, if $n = 2k + 1$,

$$e_{2k+1} \mapsto i T \otimes T \otimes \dots \otimes T \otimes T \otimes T.$$

The vectors

$$u_{+1} = \frac{1}{\sqrt{2}}(1, -i) \quad \text{and} \quad u_{-1} = \frac{1}{\sqrt{2}}(1, i),$$

form a unitary basis of \mathbb{C}^2 with respect to the standard Hermitian product. Thus

$$\{u_{\varepsilon_1, \dots, \varepsilon_k} = u_{\varepsilon_1} \otimes \dots \otimes u_{\varepsilon_k} \mid \varepsilon_j = \pm 1, j = 1, \dots, k\},$$

is a unitary basis of $\Delta_n = \mathbb{C}^{2^k}$ with respect to the naturally induced Hermitian product.

The Clifford multiplication is defined by

$$\begin{aligned}\mu_n : \mathbb{R}^n \otimes \Delta_n &\longrightarrow \Delta_n \\ x \otimes \psi &\mapsto \mu_n(x \otimes \psi) = x \cdot \psi := \kappa(x)(\psi).\end{aligned}$$

Additionally, the maps

$$\alpha \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} -\bar{z}_2 \\ \bar{z}_1 \end{pmatrix}, \quad \beta \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \end{pmatrix},$$

define quaternionic and real structures, respectively, on \mathbb{C}^2 . Using α and β , real or quaternionic structures γ_n are built on $\Delta_n = (\mathbb{C}^2)^{\otimes [n/2]}$, for $n \geq 2$, as follows

$$\begin{aligned}\gamma_n &= (\alpha \otimes \beta)^{\otimes 2k} && \text{if } n = 8k, 8k+1 && \text{(real),} \\ \gamma_n &= \alpha \otimes (\beta \otimes \alpha)^{\otimes 2k} && \text{if } n = 8k+2, 8k+3 && \text{(quaternionic),} \\ \gamma_n &= (\alpha \otimes \beta)^{\otimes 2k+1} && \text{if } n = 8k+4, 8k+5 && \text{(quaternionic),} \\ \gamma_n &= \alpha \otimes (\beta \otimes \alpha)^{\otimes 2k+1} && \text{if } n = 8k+6, 8k+7 && \text{(real).}\end{aligned}$$

The Spin group $Spin(n) \subset Cl_n$ is the subset

$$Spin(n) = \{x_1 x_2 \cdots x_{2l-1} x_{2l} \mid x_j \in \mathbb{R}^n, |x_j| = 1, l \in \mathbb{N}\},$$

endowed with the product of the Clifford algebra. It is a Lie group and its Lie algebra is

$$\mathfrak{spin}(n) = \text{span}\{e_i e_j \mid 1 \leq i < j \leq n\}.$$

Recall that the Spin group $Spin(n)$ is the universal double cover of $SO(n)$, $n \geq 3$. For $n = 2$ we consider $Spin(2)$ to be the connected double cover of $SO(2)$. The covering map will be denoted by

$$\lambda_n : Spin(n) \rightarrow SO(n).$$

Its differential is given by $(\lambda_n)_*(e_i e_j) = 2E_{ij}$, where $E_{ij} = e_i^* \otimes e_j - e_j^* \otimes e_i$ is the standard basis of the skew-symmetric matrices and e^* denotes the metric dual of the vector e . Furthermore, we will abuse the notation and also denote by λ_n the induced representation on $\bigwedge^* \mathbb{R}^n$.

The restriction of κ to $Spin(n)$ defines the Lie group representation

$$\kappa_n : Spin(n) \longrightarrow GL(\Delta_n),$$

which is special unitary. We have the corresponding Lie algebra representation

$$\kappa_{n*} : \mathfrak{spin}(n) \longrightarrow \mathfrak{gl}(\Delta_n).$$

Remark. For the sake of notation we will set

$$\begin{aligned}SO(0) &= \{1\}, & SO(1) &= \{1\}, \\ Spin(0) &= \{\pm 1\}, & Spin(1) &= \{\pm 1\},\end{aligned}$$

and

$$\Delta_0 = \Delta_1 = \mathbb{C}$$

a trivial 1-dimensional representation.

The Clifford multiplication μ_n is skew-symmetric with respect to the Hermitian product

$$\langle x \cdot \psi_1, \psi_2 \rangle = -\langle \psi_1, x \cdot \psi_2 \rangle,$$

is $Spin(n)$ -equivariant and can be extended to a $Spin(n)$ -equivariant map

$$\begin{aligned}\mu_n : \bigwedge^* (\mathbb{R}^n) \otimes \Delta_n &\longrightarrow \Delta_n \\ \omega \otimes \psi &\mapsto \omega \cdot \psi.\end{aligned}$$

2.2 Twisted Spin groups

Consider the following groups:

- By using the unit complex numbers $U(1)$, the Spin group can be twisted [10]

$$Spin^c(n) = (Spin(n) \times U(1)) / \{\pm(1, 1)\} = Spin(n) \times_{\mathbb{Z}_2} U(1),$$

with Lie algebra

$$\mathfrak{spin}^c(n) = \mathfrak{spin}(n) \oplus i\mathbb{R}.$$

- By using $Spin^c(r)$ define

$$\begin{aligned} Spin^{c,r}(n) &= (Spin(n) \times Spin^c(r)) / \{\pm(1, 1)\} \\ &= Spin(n) \times_{\mathbb{Z}_2} Spin^c(r), \end{aligned}$$

where $r \in \mathbb{N}$, whose Lie algebra is

$$\mathfrak{spin}^c(n) = \mathfrak{spin}(n) \oplus \mathfrak{spin}(r) \oplus i\mathbb{R}.$$

It fits into the exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow Spin^{c,r}(n) \xrightarrow{\lambda_{n,r,2}} SO(n) \times SO(r) \times U(1) \longrightarrow 1,$$

where

$$(\lambda_{n,r,2})[g, [h, z]] = (\lambda_n(g), \lambda_r(h), z^2).$$

- Let $\widehat{Spin^c(r)}$ denote the standard copy of $Spin^c(r)$ in $Spin^{c,r}(n)$ given by elements of the form $[1, [h, z]]$ where $h \in Spin(r)$ and $z \in U(1)$.

Remark. For $r = 0, 1$, $Spin^{c,r}(n) = Spin^c(n)$.

Lemma 2.1. [12] *Let $r \in \mathbb{N}$. There exists a monomorphism $h : U(m) \times SO(r) \hookrightarrow Spin^{c,r}(2m+r)$ such that the following diagram commutes*

$$\begin{array}{ccc} & & Spin^{c,r}(2m+r) \\ & \nearrow & \downarrow \\ U(m) \times SO(r) & \longrightarrow & SO(2m+r) \times SO(r) \times U(1) \end{array}$$

□

Lemma 2.2. *Let $n \geq 3$.*

- For $r \geq 3$

$$\pi_1(Spin^{c,r}(n)) = \mathbb{Z}_2 \oplus \mathbb{Z}.$$

- For $r = 2$

$$\pi_1(Spin^{c,2}(n)) = \mathbb{Z} \oplus \mathbb{Z}.$$

Proof. For $r \geq 3$, consider the universal cover

$$\begin{array}{c} Spin(n) \times Spin(r) \times \mathbb{R} \\ \downarrow \\ Spin^{c,r}(n) \end{array}$$

The preimage of $[1, [1, 1]] \in Spin^{c,r}(n)$ is

$$\langle (-1, -1, 0), (1, -1, 1) \rangle \subset Spin(n) \times Spin(r) \times \mathbb{R}.$$

For $r = 2$, consider the universal cover

$$\begin{array}{c} Spin(n) \times \mathbb{R} \times \mathbb{R} \\ \downarrow \\ Spin^{c,2}(n) \end{array}$$

The preimage of $[1, [1, 1]] \in Spin^{c,2}(n)$ is

$$\langle (-1, 1, 0), (-1, 0, 1) \rangle \subset Spin(n) \times \mathbb{R} \times \mathbb{R}.$$

□

2.3 Twisted Spin representations

Consider the following twisted representations:

- The Spin representation Δ_n extends to a representation of $Spin^c(n)$ by letting

$$\begin{array}{ccc} Spin^c(n) & \longrightarrow & GL(\Delta_n) \\ [g, z] & \mapsto & z\kappa_n(g) =: zg. \end{array}$$

- The twisted $Spin^{c,r}(n)$ representation is given by

$$\begin{array}{ccc} \kappa_n^{c,r} : Spin^{c,r}(n) & \longrightarrow & GL(\Delta_r \otimes \Delta_n) \\ [g, [h, z]] & \mapsto & z\kappa_r(h) \otimes \kappa_n(g) =: zh \otimes g \end{array}$$

which is also unitary with respect to the natural Hermitian metric.

- For $r = 0, 1$, the twisted Spin representation is simply the $Spin^c(n)$ representation Δ_n .

We will also need the $Spin^{c,r}(n)$ -equivariant map

$$\begin{array}{ccc} \mu_r \otimes \mu_n : \left(\bigwedge^* \mathbb{R}^r \otimes_{\mathbb{R}} \bigwedge^* \mathbb{R}^n \right) \otimes_{\mathbb{R}} (\Delta_r \otimes \Delta_n) & \longrightarrow & \Delta_r \otimes \Delta_n \\ (w_1 \otimes w_2) \otimes (\psi \otimes \varphi) & \mapsto & (w_1 \otimes w_2) \cdot (\psi \otimes \varphi) = (w_1 \cdot \psi) \otimes (w_2 \cdot \varphi). \end{array}$$

2.4 Skew-symmetric 2-forms and endomorphisms associated to twisted spinors

We will often write f_{kl} for the Clifford product $f_k f_l$.

Definition 2.1. [9] Let $r \geq 2$, $\phi \in \Delta_r \otimes \Delta_n$, $X, Y \in \mathbb{R}^n$, $(f_1 \dots, f_r)$ an orthonormal basis of \mathbb{R}^r and $1 \leq k, l \leq r$.

- Define the real 2-forms associated to the spinor ϕ by

$$\eta_{kl}^\phi(X, Y) = \text{Re} \langle X \wedge Y \cdot \kappa_{r*}(f_k f_l) \cdot \phi, \phi \rangle.$$

- Define the antisymmetric endomorphisms $\hat{\eta}_{kl}^\phi \in \text{End}^-(\mathbb{R}^n)$ by

$$X \mapsto \hat{\eta}_{kl}^\phi(X) := (X \lrcorner \eta_{kl}^\phi)^\sharp,$$

where $X \in \mathbb{R}^n$, \lrcorner denotes contraction and $^\sharp$ denotes metric dualization from 1-forms to vectors.

Observe that $\eta_{kl}^\phi = (\delta_{kl} - 1)\eta_{lk}^\phi$. If $k \neq l$ then the imaginary part of η_{kl}^ϕ vanishes, so we can write

$$\eta_{kl}^\phi(X, Y) = \langle X \wedge Y \cdot \kappa_{r*}(f_k f_l) \cdot \phi, \phi \rangle.$$

Lemma 2.3. [9] Any spinor $\phi \in \Delta_r \otimes \Delta_n$, $r \geq 2$, defines two maps (extended by linearity)

$$\begin{aligned} \bigwedge^2 \mathbb{R}^r &\longrightarrow \bigwedge^2 \mathbb{R}^n \\ f_{kl} &\mapsto \eta_{kl}^\phi \end{aligned}$$

and

$$\begin{aligned} \bigwedge^2 \mathbb{R}^r &\longrightarrow \text{End}(\mathbb{R}^n) \\ f_{kl} &\mapsto \hat{\eta}_{kl}^\phi. \end{aligned}$$

2.5 Twisted partially pure spinors

In order to simplify the statements, we will consider the twisted Spin representation

$$\Sigma_r \otimes \Delta_n \subseteq \Delta_r \otimes \Delta_n.$$

where

$$\Sigma_r = \begin{cases} \Delta_r & \text{if } r \text{ is odd,} \\ \Delta_r^+ & \text{if } r \text{ is even,} \end{cases}$$

and $n, r \in \mathbb{N}$.

Definition 2.2. [12] Let (f_1, \dots, f_r) be an orthonormal frame of \mathbb{R}^r . A unit-length spinor $\phi \in \Sigma_r \otimes \Delta_n$, $r < n$, is called a twisted partially pure spinor if

- there exists a $(n - r)$ -dimensional subspace $V^\phi \subset \mathbb{R}^n$ such that for every $X \in V^\phi$, there exists a $Y \in V^\phi$ such that

$$X \cdot \phi = i Y \cdot \phi.$$

- it satisfies the equations

$$\begin{aligned} (\eta_{kl}^\phi + \kappa_{r*}(f_k f_l)) \cdot \phi &= 0, \\ \langle \kappa_{r*}(f_k f_l) \cdot \phi, \phi \rangle &= 0, \end{aligned}$$

for all $1 \leq k < l \leq r$.

- If $r = 4$, it also satisfies the condition

$$\langle \kappa(f_1 f_2 f_3 f_4) \cdot \phi, \phi \rangle = 0.$$

Let $(e_1, \dots, e_{2m}, e_{2m+1}, \dots, e_{2m+r})$ and (f_1, \dots, f_r) be orthonormal frames of \mathbb{R}^{2m+r} and \mathbb{R}^r respectively. Consider the decomposition

$$\Delta_{2m+r} = \Delta_r \otimes \Delta_{2m}^+ \oplus \Delta_r \otimes \Delta_{2m}^-,$$

corresponding to the decomposition

$$\mathbb{R}^{2m+r} = \text{span}\{e_1, \dots, e_{2m}\} \oplus \text{span}\{e_{2m+1}, \dots, e_{2m+r}\}.$$

Let

$$\varphi_0 = u_{1,\dots,1} \in \Delta_{2m}^+,$$

and

$$\{v_{\varepsilon_1, \dots, \varepsilon_{[r/2]}} \mid (\varepsilon_1, \dots, \varepsilon_{[r/2]}) \in \{\pm 1\}^{[r/2]}\}$$

be the unitary basis of the twisting factor $\Delta_r = \Delta(\text{span}(f_1, \dots, f_r))$ which contains Σ_r . Let us define the standard twisted partially pure spinor $\phi_0 \in \Sigma_r \otimes \Delta_r \otimes \Delta_{2m}^+$ by

$$\phi_0 = \begin{cases} \frac{1}{\sqrt{2^{[r/2]}}} \left(\sum_{I \in \{\pm 1\}^{\times [r/2]}} v_I \otimes \gamma_r(u_I) \right) \otimes \varphi_0 & \text{if } r \text{ is odd,} \\ \frac{1}{\sqrt{2^{[r/2]-1}}} \left(\sum_{I \in [\{\pm 1\}^{\times [r/2]}]_+} v_I \otimes \gamma_r(u_I) \right) \otimes \varphi_0 & \text{if } r \text{ is even,} \end{cases} \quad (1)$$

where the elements of $[\{\pm 1\}^{\times [r/2]}]_+$ contain an even number of (-1) .

We collect properties of partially pure spinors [12] in the following proposition.

Proposition 2.1. *Let $\phi \in \Sigma_r \otimes \Delta_n$ be a partially pure spinor.*

- *The definition of partially pure spinor does not depend on the choice of orthonormal basis of \mathbb{R}^r .*
- *There exists an orthogonal complex structure on V^ϕ and $n - r \equiv 0 \pmod{2}$.*
- *If $r \geq 2$,*

$$\text{span}\{\hat{\eta}_{kl}^\phi \in \text{End}^-(\mathbb{R}^n) \mid 1 \leq k < l \leq r\} \cong \mathfrak{so}(r).$$

□

2.6 Certain homogeneous spaces

In this subsection, we present certain homogeneous spaces which will provide examples for various results in the following sections.

Consider the partial flag manifold

$$\mathcal{G}_{m,s,r} = \frac{SO(2m+s+r)}{U(m) \times SO(s) \times SO(r)}$$

We will decompose the Lie algebra $\mathfrak{so}(2m+s+r)$ according to the natural inclusions

$$U(m) \times SO(s) \times SO(r) \subset SO(2m) \times SO(s) \times SO(r) \subset SO(2m+s+r).$$

Note that

$$\begin{aligned} \mathfrak{so}(2m+s+r) &= \bigwedge^2 \mathbb{R}^{2m+s+r} \\ &= \bigwedge^2 (\mathbb{R}^{2m} \oplus \mathbb{R}^s \oplus \mathbb{R}^r) \\ &= \bigwedge^2 \mathbb{R}^{2m} \oplus \bigwedge^2 \mathbb{R}^s \oplus \bigwedge^2 \mathbb{R}^r \oplus \mathbb{R}^{2m} \otimes \mathbb{R}^s \oplus \mathbb{R}^{2m} \otimes \mathbb{R}^r \oplus \mathbb{R}^s \otimes \mathbb{R}^s \end{aligned}$$

$$\begin{aligned}
&= \mathfrak{so}(2m) \oplus \mathfrak{so}(s) \oplus \mathfrak{so}(r) \oplus \mathbb{R}^{2m} \otimes \mathbb{R}^s \oplus \mathbb{R}^{2m} \otimes \mathbb{R}^r \oplus \mathbb{R}^s \otimes \mathbb{R}^r \\
\mathfrak{so}(2m) \otimes \mathbb{C} &= \bigwedge^2 (\mathbb{C}^m \oplus \overline{\mathbb{C}^m}) \\
&= \bigwedge^2 \mathbb{C}^m \oplus \mathbb{C}^m \otimes \overline{\mathbb{C}^m} \oplus \bigwedge^2 \overline{\mathbb{C}^m} \\
&= [[\bigwedge^2 \mathbb{C}^m]] \otimes \mathbb{C} \oplus \mathfrak{u}(m) \otimes \mathbb{C}, \\
\mathbb{R}^{2m} &= [[\mathbb{C}^m]],
\end{aligned}$$

where the symbol $[[\mathbb{C}^m]]$ denotes the underlying real vector space \mathbb{R}^{2m} of \mathbb{C}^m carrying a complex structure. Thus

$$\mathfrak{so}(2m + s + r) = \mathfrak{u}(m) \oplus \mathfrak{so}(s) \oplus \mathfrak{so}(r) \oplus \left([[\bigwedge^2 \mathbb{C}^m]] \oplus [[\mathbb{C}^m]] \otimes \mathbb{R}^s \oplus [[\mathbb{C}^m]] \otimes \mathbb{R}^r \oplus \mathbb{R}^s \otimes \mathbb{R}^r \right)$$

and the tangent space of $\mathcal{G}_{m,s,r}$ decomposes as follows

$$T_{\text{Id}} \mathcal{G}_{m,s,r} \cong [[\bigwedge^2 \mathbb{C}^m]] \oplus [[\mathbb{C}^m]] \otimes \mathbb{R}^s \oplus [[\mathbb{C}^m]] \otimes \mathbb{R}^r \oplus \mathbb{R}^s \otimes \mathbb{R}^r.$$

This gives the isotropy representation

$$\begin{aligned}
U(m) \times SO(s) \times SO(r) &\longrightarrow SO(T_{\text{Id}} \mathcal{G}_{m,s,r}) \\
(A, B, C) &\mapsto \begin{pmatrix} [[\bigwedge^2 A]] & & & \\ & [[A]] \otimes B & & \\ & & [[A]] \otimes C & \\ & & & B \otimes C \end{pmatrix},
\end{aligned}$$

where $\bigwedge^2 A$ denotes the linear transformation induced by A on $\bigwedge^2 \mathbb{C}^m$, $[[A]]$ the transformation A viewed as a real linear transformation on $[[\mathbb{C}^m]] = \mathbb{R}^{2m}$, and $B \otimes C$ the induced transformation on $\mathbb{R}^s \otimes \mathbb{R}^r$ (i.e. the Kronecker product of B and C).

3 Doubly twisted Spin structures

In this section, we introduce the (doubly) twisted Spin structures we need to carry out our spinorial characterization of CR structures, and the corresponding twisted Dirac operator and Laplacian. We deduce some topological conditions on manifolds that support such structures, a Schrödinger-Lichnerowicz type formula, and give some Bochner-type arguments.

Definition 3.1. *Let M be an oriented n -dimensional Riemannian manifold, $P_{SO(M)}$ be its principal bundle of orthonormal frames and $r \in \mathbb{N}$. A $\text{Spin}^{c,r}(n)$ structure on M consists of an auxiliary principal $SO(r)$ bundle $P_{SO(r)}$, an auxiliary principal $U(1)$ bundle $P_{U(1)}$ and a principal $\text{Spin}^{c,r}(n)$ bundle $P_{\text{Spin}^{c,r}(n)}$ together with an equivariant $2:1$ covering map*

$$\Lambda : P_{\text{Spin}^{c,r}(n)} \longrightarrow P_{SO(M)} \tilde{\times} P_{SO(r)} \tilde{\times} P_{U(1)},$$

where $\tilde{\times}$ denotes the fibered product, such that $\Lambda(pg) = \Lambda(p)(\lambda_{n,r,2})(g)$ for all $p \in P_{\text{Spin}^{c,r}(n)}$ and $g \in \text{Spin}^{c,r}(n)$, where $\lambda_{n,r,2} : \text{Spin}^{c,r}(n) \longrightarrow SO(n) \times SO(r) \times U(1)$ denotes the canonical 2-fold cover.

A n -dimensional Riemannian manifold M admitting a $\text{Spin}^{c,r}(n)$ structure will be called a $\text{Spin}^{c,r}$ manifold.

Remark. A $\text{Spin}^{c,r}$ manifold with trivial $P_{SO(r)}$ and $P_{U(1)}$ auxiliary bundles is a Spin manifold. On the other hand, we have the following:

- Any Spin manifold admits a $\text{Spin}^{c,r}$ structure with trivial $P_{SO(r)}$ and $P_{U(1)}$ auxiliary bundles via the inclusion $\text{Spin}(n) \subset \text{Spin}^{c,r}(n)$.
- Any Spin^c manifold admits a $\text{Spin}^{c,r}$ structure with trivial $P_{SO(r)}$ auxiliary bundle via the inclusion $\text{Spin}^c(n) \subset \text{Spin}^{c,r}(n)$.
- Any Spin^r manifold (cf. [9]) admits a $\text{Spin}^{c,r}$ structure with trivial $P_{U(1)}$ auxiliary bundle via the inclusion $\text{Spin}^r(n) \subset \text{Spin}^{c,r}(n)$.

3.1 Existence of $\text{Spin}^{c,r}$ structures

We will characterize the existence of a $\text{Spin}^{c,r}$ structure in terms of a Spin structure.

Proposition 3.1. [10, p. 47] *Let $G \subset SO(N)$ be a connected compact Lie subgroup with*

$$\pi_1(SO(N)/G) = \{0\}.$$

A G -principal bundle Q over a connected CW-complex X has a Spin structure if and only if there exists a homomorphism $f : \pi_1(Q) \rightarrow \pi_1(SO(N))$ for which the diagram

$$\begin{array}{ccc} \pi_1(G) & \xrightarrow{i_{\#}} & \pi_1(SO(n)) \\ h \downarrow & \nearrow f & \\ \pi_1(Q) & & \end{array}$$

commutes.

□

By setting $N = n + r + 2$, $G = SO(n) \times SO(r) \times U(1)$, $Q = P_{SO(M)} \tilde{\times} P_{SO(r)} \tilde{\times} P_{U(1)}$ and considering the natural inclusion of $SO(n) \times SO(r) \times U(1) \subset SO(n + r + 2)$ we have that

$$\pi_1 \left(\frac{SO(n + r + 2)}{SO(n) \times SO(r) \times U(1)} \right) = \{0\}.$$

Corollary 3.1. *The bundle $P_{SO(M)} \tilde{\times} P_{SO(r)} \tilde{\times} P_{U(1)}$ over M has a Spin structure if and only if there exists a homomorphism $f : \pi_1(Q) \rightarrow \pi_1(SO(n + r + 2))$ for which the diagram*

$$\begin{array}{ccc} \pi_1(SO(n) \times SO(r) \times U(1)) & \xrightarrow{i_{\#}} & \pi_1(SO(n + r + 2)) \\ h \downarrow & \nearrow f & \\ \pi_1(P_{SO(M)} \tilde{\times} P_{SO(r)} \tilde{\times} P_{U(1)}) & & \end{array}$$

commutes.

□

Lemma 3.1. *For $r \geq 2$,*

$$\ker(i_{\#}) = (\lambda_n \times \lambda_r \times \lambda_2)_{\#}(\pi_1(\text{Spin}^{c,r}(n))).$$

Proof. Using additive notation, we have for $r \geq 3$

$$\begin{aligned} \pi_1(SO(n) \times SO(r) \times U(1)) &\cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z} & \xrightarrow{i_{\#}} & \pi_1(SO(n + r + 2)) \cong \mathbb{Z}_2 \\ (a, b, c) & & \mapsto & a + b + c \pmod{2}, \end{aligned}$$

and for $r = 2$

$$\begin{aligned} \pi_1(SO(n) \times SO(2) \times U(1)) &\cong \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{i\#} \pi_1(SO(n+4)) \cong \mathbb{Z}_2 \\ (a, b, c) &\mapsto a + b + c \pmod{2}. \end{aligned}$$

For $r \geq 3$, we see that $\pi_1(\text{Spin}^{c,r}(n)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}$, as described in Lemma 2.2, is mapped as follows

$$\begin{aligned} \pi_1(\text{Spin}^{c,r}(n)) &\xrightarrow{(\lambda_n \times \lambda_r \times \lambda_2)\#} \pi_1(SO(n) \times SO(r) \times U(1)) \xrightarrow{i\#} \pi_1(SO(n+r+2)) \\ (a, b) &\mapsto (a, a+b, b) \mapsto 2a+2b = 0 \pmod{2}. \end{aligned}$$

For $r = 2$, we see that $\pi_1(\text{Spin}^{c,2}(n)) \cong \mathbb{Z} \oplus \mathbb{Z}$, as described in Lemma 2.2, is mapped as follows

$$\begin{aligned} \pi_1(\text{Spin}^{c,2}(n)) &\xrightarrow{(\lambda_n \times \lambda_2 \times \lambda_2)\#} \pi_1(SO(n) \times SO(2) \times U(1)) \xrightarrow{i\#} \pi_1(SO(n+4)) \\ (a, b) &\mapsto (a+b, a, b) \mapsto 2a+2b = 0 \pmod{2}. \end{aligned}$$

□

Proposition 3.2. *M admits an $SO(r) \times SO(2)$ -principal bundle $P_{SO(r) \times SO(2)}$ such that the fibre product $Q = P_{SO(n)} \tilde{\times} P_{SO(r) \times SO(2)}$ has a Spin structure if and only if M has a $\text{Spin}^{c,r}$ structure.*

Proof. If M has a $\text{Spin}^{c,r}$ structure then $P_{SO(2)} := P/\text{Spin}^r(n)$ and $P_{SO(r)} := P/\text{Spin}^c(n)$ are $SO(2)$ and $SO(r)$ principal bundles respectively, so that $P_{SO(r) \times SO(2)} := P_{SO(r)} \tilde{\times} P_{SO(2)}$ is a $SO(r) \times SO(2)$ principal bundle over M . Now, there exists an injective homomorphism \tilde{i} which makes the diagram

$$\begin{array}{ccc} \text{Spin}^{c,r}(n) & \xrightarrow{\tilde{i}} & \text{Spin}(n+r+2) \\ \downarrow & & \downarrow \\ SO(n) \times SO(r) \times SO(2) & \xrightarrow{i} & SO(n+r+2) \end{array}$$

commute. From this we obtain a Spin structure for Q in the sense of Corollary 3.1.

Conversely, let $\lambda = \lambda_n \times \lambda_r \times \lambda_2$ and $F = SO(n) \times SO(r) \times U(1)$. According to Corollary 3.1, due to the existence of f , $H = \ker(f) \subset \pi_1(Q)$ is a subgroup of index 2. Therefore, there exists a double covering space $\Lambda : P_{\text{Spin}^{c,r}(n)} \rightarrow Q$ corresponding to H . Let $\mu : Q \times F \rightarrow Q$ be the action of F in Q and consider the composition of induced maps on fundamental groups

$$\pi_1(P_{\text{Spin}^{c,r}(n)} \times \text{Spin}^{c,r}(n)) \xrightarrow{(\Lambda \times \lambda)\#} \pi_1(Q \times F) \xrightarrow{\mu\#} \pi_1(Q).$$

If $(\sigma, \tau) \in \pi_1(P_{\text{Spin}^{c,r}(n)}) \times \pi_1(\text{Spin}^{c,r}(n))$, by means of the inclusion h ,

$$\begin{aligned} \mu\# \circ (\Lambda \times \lambda)\#(\sigma, \tau) &= \Lambda\#(\sigma) \lambda\#(\tau) \\ &= \Lambda\#(\sigma) * h(\lambda\#(\tau)) \end{aligned}$$

where $*$ denotes product in the relevant fundamental group. We know that

$$\Lambda\#(\sigma) \in H \quad \text{and} \quad f(h(\lambda\#(\tau))) = i\#(\lambda\#(\tau)) = 0$$

by Lemma 3.1 and Corollary 3.1. Thus, $h(\lambda\#(\tau)) \in H$ and $\Lambda\#(\sigma) * h(\lambda\#(\tau)) \in H$. Hence, there exists a lift $\tilde{\mu} : P_{\text{Spin}^{c,r}(n)} \times \text{Spin}^{c,r}(n) \rightarrow P_{\text{Spin}^{c,r}(n)}$ which gives the equivariance in Definition 3.1. □

Now, we will derive a condition for a simply connected manifold to have a “non-reducible” $\text{Spin}^{c,r}$ structure, i.e. a $\text{Spin}^{c,r}$ structure which does not come from a Spin , nor a Spin^c , nor Spin^r structure.

Proposition 3.3. *Let M be simply connected and Q its $SO(n)$ -principal bundle of orthonormal frames. The following are equivalent*

1. Q has a $\text{Spin}^{c,r}$ structure but does not have a Spin , nor a Spin^c , nor a Spin^r structure.
2. There exists a $SO(r) \times SO(2)$ bundle P_1 over X such that in the long exact sequence

$$\cdots \longrightarrow \pi_2(X) \xrightarrow{\partial} \pi_1(SO(n) \times SO(r) \times SO(2)) \xrightarrow{h} \pi_1(Q \tilde{\times} P_1) \longrightarrow \pi_1(X) = 0,$$

$$\text{Im}(\partial) \cong \langle (1, 0, p), (0, 1, p) \rangle \subset \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z} \text{ with } p \text{ odd.}$$

Proof. If (P, Λ) is a $\text{Spin}^{c,r}$ structure on Q then $P_{SO(2)} := P/\text{Spin}^r(n)$ and $P_{SO(r)} := P/\text{Spin}^c(n)$ are $SO(2)$ and $SO(r)$ principal bundles respectively, so that $P_1 := P_{SO(r)} \tilde{\times} P_{SO(2)}$ is a $SO(r) \times SO(2)$ principal bundle over X . Now, by Proposition 3.2, the fibre product $Q \times P_1$ has a Spin structure. By Corollary 3.1, this means that there exists a map $f : \pi_1(Q \tilde{\times} P_1) \rightarrow \pi_1(SO(n+r+2))$ such that the diagram

$$\begin{array}{ccc} \pi_1(SO(n) \times SO(r) \times SO(2)) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z} & \xrightarrow{i\#} & \pi_1(SO(n+r+2)) = \mathbb{Z}_2 \\ \downarrow h & \nearrow f & \\ \pi_1(Q \tilde{\times} P_1) & & \end{array}$$

commutes. Now, if Q does not have a Spin structure then we have $\pi_1(Q) = 0$ in the following commutative diagram

$$\begin{array}{ccccccc} & & & & \vdots & & (2) \\ & & & & \downarrow & & \\ & & & & \pi_2(X) & & \\ & & & & \downarrow \partial & & \\ & & & & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z} & & \\ & \nearrow j\# & & & \downarrow h & & \\ \mathbb{Z}_2 \oplus \mathbb{Z} = \pi_1(SO(r) \times SO(2)) & \xrightarrow{k} & \pi_1(Q \tilde{\times} P_1) & \longrightarrow & \pi_1(Q) & \longrightarrow & 0 \\ & & \downarrow & & & & \\ & & \pi_1(X) & = & 0. & & \end{array}$$

Thus, k is onto and

$$(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z})/\text{Im}(\partial) \cong \pi_1(Q \tilde{\times} P_1) = h(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}) = k(\mathbb{Z}_2 \oplus \mathbb{Z}).$$

Now, we will describe the group $K = \pi_1(Q \tilde{\times} P_1)$. It depends on the nontrivial elements $h(1, 0, 0) = \alpha$, $h(0, 1, 0) = \beta$ and $h(0, 0, 1) = \gamma$. First, we have $K = \langle \beta, \gamma \rangle$, so that $\alpha = a\beta + b\gamma$ for some integers a, b . Since α and β have order two in K

$$0 = 2\alpha = 2a\beta + 2b\gamma = 2b\gamma,$$

and K is a finite group. Now,

- (i) If $\beta \in \langle \gamma \rangle$ then γ has order $2p$, $K \cong \mathbb{Z}_{2p}$ and $\alpha = \beta = p\gamma$. Since there is only one nontrivial map $f : \mathbb{Z}_{2p} \rightarrow \mathbb{Z}_2$, $f \circ h = i_{\#}$ if and only if p is odd.
- (ii) If $\beta \notin \langle \gamma \rangle$ then $K \cong \mathbb{Z}_2 \oplus \mathbb{Z}_d$. If d is odd, $f : \mathbb{Z}_d \rightarrow \mathbb{Z}_2$ must be trivial and $(f \circ h)(0, 0, 1) = 0$ which gives us no $\text{Spin}^{c,r}$ structure. If $d = 2p$, then

$$\alpha = \beta, \quad \text{or} \quad \alpha = p\gamma \quad \text{or} \quad \alpha = \beta + p\gamma.$$

In order to have $i_{\#} = f \circ h$ and, therefore, the existence of the $\text{Spin}^{c,r}$ structure, if $\alpha = p\gamma$ then p must be odd, and if $\alpha = \beta + p\gamma$ then p must be even.

Now, we are going to rule out the three options in (ii). Note that $K' = \pi_1(Q \tilde{\times} P_{SO(r)} \tilde{\times} P_{SO(2)})$ and the $SO(2)$ fibre bundle $Q \tilde{\times} P_{SO(r)} \tilde{\times} P_{SO(2)} \rightarrow Q \tilde{\times} P_{SO(r)}$ gives the commutative diagram

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & (3) \\
 & & \downarrow & & \downarrow & & \\
 & & \pi_2(X) & & \pi_2(X) & & \\
 & & \downarrow & & \downarrow & & \\
 \cdots \longrightarrow & \mathbb{Z} = \pi_1(SO(2)) & \longrightarrow & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \longrightarrow 0 \\
 & \nearrow & & \downarrow & & \downarrow & \\
 \cdots \longrightarrow & \mathbb{Z} = \pi_1(SO(2)) & \longrightarrow & \pi_1(Q \tilde{\times} P_{SO(r)} \tilde{\times} P_{SO(2)}) & \longrightarrow & \pi_1(Q \tilde{\times} P_{SO(r)}) & \longrightarrow 0 \\
 & & & \downarrow & & \downarrow & \\
 & & & \pi_1(X) = 0 & & \pi_1(X) = 0 &
 \end{array}$$

- If $K' = \mathbb{Z}_2 \oplus \mathbb{Z}_{2p} = \langle \beta, \gamma \rangle$ with $\alpha = \beta$ then, by exactness of the diagram, $\pi_1(Q \tilde{\times} P_{SO(r)}) = (\mathbb{Z}_2 \oplus \mathbb{Z}_{2p}) / \langle \gamma \rangle = \mathbb{Z}_2 \cong \langle \beta \rangle$, which gives us a Spin^r structure.
- The same happens if $\alpha = \beta + p\gamma$. The quotient is isomorphic to \mathbb{Z}_2 , whose equivalence classes are

$$\{(0, 0), (0, 1), \dots, (0, 2p - 1)\} \quad \text{and} \quad \{(1, 0), \dots, (1, 2p - 1)\},$$

where $\gamma = (0, 1)$ belongs to the first one, and $\alpha = (1, p)$ and $\beta = (1, 0)$ belong to the second one. In other words, α and β are mapped to the nontrivial class and we have a Spin^r structure.

- Now if $K' = \mathbb{Z}_2 \oplus \mathbb{Z}_{2p}$ with p odd and $\alpha = p\gamma$, the $SO(r)$ fibre bundle $Q \tilde{\times} P_{SO(r)} \tilde{\times} P_{SO(2)} \rightarrow$

$Q \tilde{\times} P_{SO(2)}$ gives the commutative diagram

$$\begin{array}{ccccccc}
& & \vdots & & \vdots & & \\
& & \downarrow & & \downarrow & & \\
& & \pi_2(X) & & \pi_2(X) & & \\
& & \downarrow & & \downarrow & & \\
\cdots \longrightarrow & \mathbb{Z}_2 = \pi_1(SO(r)) & \longrightarrow & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z}_2 \oplus \mathbb{Z} & \longrightarrow 0 \\
& \nearrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & \mathbb{Z}_2 = \pi_1(SO(r)) & \longrightarrow & \pi_1(Q \tilde{\times} P_{SO(r)} \tilde{\times} P_{SO(2)}) & \longrightarrow & \pi_1(Q \tilde{\times} P_{SO(2)}) & \longrightarrow 0 \\
& & & \downarrow & & \downarrow & \\
& & & \pi_1(X) = 0 & & \pi_1(X) = 0 &
\end{array}$$

so that $\pi_1(Q \tilde{\times} P_{SO(2)}) = \mathbb{Z}_2 \oplus \mathbb{Z}_{2p} / \langle \beta \rangle = \mathbb{Z}_{2p}$ with p odd, which implies the existence of a Spin^c structure.

Now we know that $K = \mathbb{Z}_{2p} = \langle \gamma \rangle$ with p odd and $\alpha = \beta = p\gamma$. This should be the same as the quotient $(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}) / \text{Im}(\partial)$ where, by exactness, $\text{Im}(\partial) = \ker(h)$. We see that the map h is given by $h(a, b, c) = ((a+b)p + c)\gamma$. The kernel of this map is given by the $(a, b, c) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}$ such that $(a+b)p + c \equiv 0 \pmod{2p}$, i.e. $(a, b, c) \in \langle (0, 1, p), (1, 0, p) \rangle$ where p is odd.

Conversely, assume $\text{Im}(\partial) = \langle (1, 0, p), (0, 1, p) \rangle$, p odd, and put this in the diagram (2). By exactness of the column, $\pi_1(Q \tilde{\times} P_1) \cong \mathbb{Z}_{2p}$. This group is generated by the non trivial element $\gamma = h(0, 0, 1)$, and we have $h(1, 0, 0) = h(0, 1, 0) = p\gamma$. Thus, $k = h \circ j_{\#}$ is onto, we have no Spin structure and the only nonzero homomorphism $f : \mathbb{Z}_{2p} \rightarrow \mathbb{Z}_2$ gives us $i_{\#} = f \circ h$, i.e. the existence of a $\text{Spin}^{c,r}$ structure.

The $SO(r)$ bundle $P_{SO(r)} = P_1/SO(2)$ fits into a commutative diagram similar to (3). By exactness, we have $\pi_1(Q \tilde{\times} P_{SO(r)}) = \{0\}$ and we cannot have a Spin^r structure. Similarly, the $SO(2)$ bundle $P_{SO(2)} = P_1/SO(r)$ fits into a similar diagram so that $\pi_1(Q \tilde{\times} P_{SO(2)}) = \mathbb{Z}_p$, and there is not map f as in Corollary 3.1 to have a Spin^c structure. Note that in the case $p = 1$ this last group is zero. \square

Example. Now we will give an example of a manifold satisfying the conditions of the previous Proposition. Let $X = G/H$ with $G = SO(2m+2+r)$ and $H = U(m) \times U(1) \times SO(r)$, $r \geq 3$. Since H is a compact connected subgroup of G and the inclusion map induces a map of fundamental groups which is onto, $\pi_1(X) = \{0\}$.

Now consider the bundle of orthonormal frames $Q = G \times_{\rho} SO(n)$ where $n = m^2 + 2mr + 3m + 2r = \dim(X)$ and

$$\rho : H \hookrightarrow G \times SO(n)$$

is given by the inclusion of H into the first factor and the isotropy representation in the second which is given by

$$\begin{aligned}
\xi : H &\longrightarrow SO(n) \\
(A, e^{i\theta}, B) &\mapsto [[\Lambda^2 A]] \oplus ([A] \otimes R_{\theta}) \oplus ([A] \otimes B) \oplus (R_{\theta} \otimes B),
\end{aligned}$$

where R_θ is the rotation in \mathbb{R}^2 by an angle of θ . This gives a fibration

$$\begin{array}{ccc} H & \hookrightarrow & G \times SO(n) \\ & & \downarrow \\ & & Q \end{array}$$

which induces the long exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_1(H) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2 \xrightarrow{\rho\#} \pi_1(G \times SO(n)) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow \pi_1(Q) \longrightarrow 0.$$

First note that the isotropy representation induces the map

$$\begin{aligned} \pi_1(H) &\longrightarrow \pi_1(SO(n)) \\ (a, b, c) &\mapsto (m-1+2+r)a + (2m+r)b + (2m+2)c \pmod{2} \\ &= (m-1+r)a + rb \pmod{2}. \end{aligned}$$

Thus,

$$\rho\#(a, b, c) = ((a+b+c) \pmod{2}, ((m-1+r)a + br) \pmod{2}).$$

Note that $(m-1+r)a + br \equiv 0 \pmod{2}$ if and only if r is even and m is odd. So, by exactness, $\pi_1(Q) = \mathbb{Z}_2$ (Q has a Spin structure) if and only if r is even and m is odd.

Let m and r be even and consider

$$\begin{aligned} \sigma : H &\longrightarrow (G \times SO(n)) \times U(1) \times SO(r) \\ (A, e^{i\theta}, B) &\mapsto (\rho(A, e^{i\theta}, B), e^{i\theta}, B) \end{aligned}$$

We have the fibration $H \hookrightarrow G \times SO(n) \times U(1) \times SO(r) \xrightarrow{\nu} G \times_\sigma SO(n) \times U(1) \times SO(r) = Q \tilde{\times} P_{U(1)} \tilde{\times} P_{SO(r)}$ which gives

$$\cdots \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2 \xrightarrow{\sigma\#} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}_2 \xrightarrow{\nu\#} \pi_1(Q \tilde{\times} P_{U(1)} \tilde{\times} P_{SO(r)}) \longrightarrow 0,$$

where

$$\sigma\#(a, b, c) = ((a+b+c) \pmod{2}, a \pmod{2}, b, c \pmod{2}).$$

We see that $\text{Im}(\sigma\#) = \langle (1, 1, 0, 0), (1, 0, 1, 0), (1, 0, 0, 1) \rangle = L$ is a subgroup of index two because $(1, 0, 0, 0) \notin L$ and $((1, 0, 0, 0) + L) \cup L = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}_2$. By exactness, $\pi_1(Q \tilde{\times} P_{U(1)} \tilde{\times} P_{SO(r)}) \cong \mathbb{Z}_2$.

Consider $f : \pi_1(Q \tilde{\times} P_{U(1)} \tilde{\times} P_{SO(r)}) \rightarrow \pi_1(SO(n+2+r))$ to be the only nontrivial homomorphism between these groups. Now, the inclusion of the fiber $SO(n) \times U(1) \times SO(r)$ into the fiber bundle $Q \tilde{\times} P_{U(1)} \tilde{\times} P_{SO(r)}$ is given by the inclusion j into the last three factors of $G \times SO(n) \times U(1) \times SO(r)$, followed by the projection ν . Thus, the map h in Proposition 3.3 is given by $h = \nu\# \circ j\#$.

Consider, for simplicity, $\pi_1(Q \tilde{\times} P_{U(1)} \tilde{\times} P_{SO(r)}) = \{0, 1\}$. From the explicit description of L , we can see

$$h(a, b, c) = \begin{cases} 0 & \text{if } a+b+c \equiv 0 \pmod{2}, \\ 1 & \text{if } a+b+c \equiv 1 \pmod{2}. \end{cases}$$

This means that $f \circ h$ is the same map as the inclusion of $\pi_1(SO(n) \times U(1) \times SO(r)) \hookrightarrow \pi_1(SO(n+2+r))$. By Proposition 3.3, X has a $\text{Spin}^{c,r}$ structure which does not come from either a Spin, nor a Spin^c , nor a Spin^r structure. \square

3.2 Covariant derivatives and twisted differential operators

Let M be a $\text{Spin}^{c,r}$ n -dimensional manifold, ω the Levi-Civita connection 1-form on its principal bundle of orthonormal frames $P_{SO(n)}$, θ and iA chosen connection 1-forms on the auxiliary bundles $P_{SO(r)}$ and $P_{U(1)}$ respectively. These connections forms give rise to covariant derivatives ∇ , ∇^θ and ∇^A on the associated vector bundles

$$\begin{aligned} TM &= P_{\text{Spin}^{c,r}(n)} \times_{\lambda_{n,r,2}} (\mathbb{R}^n \times \{0\} \times \{0\}), \\ F &= P_{\text{Spin}^{c,r}(n)} \times_{\lambda_{n,r,2}} (\{0\} \times \mathbb{R}^r \times \{0\}), \\ L &= P_{\text{Spin}^{c,r}(n)} \times_{\lambda_{n,r,2}} (\{0\} \times \{0\} \times \mathbb{C}), \end{aligned}$$

Furthermore, the three connections help define a connection on the twisted spinor bundle

$$S = P_{\text{Spin}^{c,r}(n)} \times_{\kappa_n^{c,r}} (\Sigma_r \otimes \Delta_n)$$

given (locally) as follows

$$\begin{aligned} \nabla^{\theta,A} : \Gamma(S) &\longrightarrow \Gamma(T^*M \otimes S) \\ \nabla^{\theta,A}(\varphi \otimes \psi) &= d(\varphi \otimes \psi) + \varphi \otimes \left[\frac{1}{2} \sum_{1 \leq i < j \leq n} \omega_{ij} \otimes e_i e_j \cdot \psi \right] \\ &\quad + \left[\frac{1}{2} \sum_{1 \leq k < l \leq r} \theta_{kl} \otimes \kappa_{r*}(f_k f_l) \cdot \varphi \right] \otimes \psi + \frac{i}{2} \varphi \otimes (A \cdot \psi), \end{aligned}$$

where $\varphi \otimes \psi \in \Gamma(S)$, (e_1, \dots, e_n) and (f_1, \dots, f_r) are a local orthonormal frames of TM and F resp., ω_{ij} , θ_{kl} and A are the corresponding local connection 1-forms for TM , F and L respectively.

From now on, we will omit the upper and lower bounds on the indices, by declaring i and j to be the indices for the frame vectors of TM , and k and l to be the indices for the frame sections of F .

Now, for any tangent vectors $X, Y \in T_x M$, the spinorial curvature is defined by

$$\begin{aligned} R^{\theta,A}(X, Y)(\varphi \otimes \psi) &= \varphi \otimes \left[\frac{1}{2} \sum_{i < j} \Omega_{ij}(X, Y) e_i e_j \cdot \psi \right] \\ &\quad + \left[\frac{1}{2} \sum_{k < l} \Theta_{kl}(X, Y) \kappa_{r*}(f_k f_l) \cdot \varphi \right] \otimes \psi + \frac{i}{2} \varphi \otimes (dA(X, Y)\psi), \end{aligned} \quad (4)$$

where

$$\Omega_{ij}(X, Y) = \langle R^M(X, Y)(e_i), e_j \rangle \quad \text{and} \quad \Theta_{kl}(X, Y) = \langle R^F(X, Y)(f_k), f_l \rangle.$$

Here R^M (resp. R^F) denotes the curvature tensor of M (resp. of F).

For X, Y vector fields and $\phi \in \Gamma(S)$ a spinor field, we have compatibility of the covariant derivative with Clifford multiplication,

$$\nabla_X^{\theta,A}(Y \cdot \phi) = (\nabla_X Y) \cdot \phi + Y \cdot \nabla_X^{\theta,A} \phi.$$

Definition 3.2. *The twisted Dirac operator is the first order differential operator $\not{D}^{\theta,A} : \Gamma(S) \longrightarrow \Gamma(S)$ defined by*

$$\not{D}^{\theta,A}(\phi) = \sum_{i=1}^n e_i \cdot \nabla_{e_i}^{\theta,A}(\phi).$$

Remark. The twisted Dirac operator $\not{D}^{\theta,A}$ is well-defined and formally self-adjoint on compact manifolds. Moreover, if $h \in C^\infty(M)$, $\phi \in \Gamma(S)$, we have

$$\not{D}^{\theta,A}(h\phi) = \text{grad}(h) \cdot \phi + h\not{D}^{\theta,A}(\phi).$$

The proofs of these facts are analogous to the ones for the Spin^c Dirac operator [10].

Definition 3.3. The twisted Spin connection Laplacian is the second order differential operator $\Delta^{\theta,A} : \Gamma(S) \rightarrow \Gamma(S)$ defined as

$$\Delta^{\theta,A}(\phi) = -\sum_{i=1}^n \nabla_{e_i}^{\theta,A} \nabla_{e_i}^{\theta,A}(\phi) - \sum_{i=1}^n \text{div}(e_i) \nabla_{e_i}^{\theta,A}(\phi).$$

3.3 A Schrödinger-Lichnerowicz-type formula

Just as in [10, 9], we have the following.

Proposition 3.4. For $X \in \Gamma(TM)$ and $\phi \in \Gamma(S)$, we have

$$\sum_{i=1}^n e_i \cdot R^{\theta,A}(X, e_i)(\phi) = -\frac{1}{2} \text{Ric}(X) \cdot \phi + \frac{1}{2} \sum_{k < l} (X \lrcorner \Theta_{kl}) \cdot \kappa_{r*}(f_k f_l) \cdot \phi + \frac{i}{2} X \lrcorner dA \cdot \phi, \quad (5)$$

where Ric denotes the Ricci tensor of M and $R^{\theta,A}$ the curvature operator of the twisted spinorial connection.

Proof. For $\phi = \varphi \otimes \psi$, by (4),

$$\begin{aligned} R^{\theta,A}(X, e_\alpha)(\varphi \otimes \psi) &= \varphi \otimes \left[\frac{1}{2} \sum_{i < j} \Omega_{ij}(X, e_\alpha) e_i e_j \cdot \psi \right] \\ &\quad + \left[\frac{1}{2} \sum_{k < l} \Theta_{kl}(X, e_\alpha) \kappa_{r*}(f_k f_l) \cdot \varphi \right] \otimes \psi + \frac{i}{2} \varphi \otimes (dA(X, e_\alpha) \psi). \end{aligned}$$

Multiply by e_α and sum over α

$$\begin{aligned} \sum_{\alpha} e_\alpha \cdot R^{\theta,A}(X, e_\alpha)(\varphi \otimes \psi) &= \varphi \otimes \left[\frac{1}{2} \sum_{\alpha} \sum_{i < j} \Omega_{ij}(X, e_\alpha) e_\alpha e_i e_j \cdot \psi \right] \\ &\quad + \frac{1}{2} [\kappa_{r*}(f_k f_l) \cdot \varphi] \otimes \sum_{k < l} \left[\sum_{\alpha} \Theta_{kl}(X, e_\alpha) e_\alpha \cdot \psi \right] \\ &\quad + \frac{i}{2} \varphi \otimes \left[\sum_{\alpha} dA(X, e_\alpha) e_\alpha \cdot \psi \right]. \end{aligned}$$

Now,

$$\begin{aligned} \frac{1}{2} \sum_{\alpha} \sum_{i < j} \Omega_{ij}(X, e_\alpha) e_\alpha e_i e_j &= -\frac{1}{2} \text{Ric}(X), \\ \kappa_{r*}(f_k f_l) \cdot \varphi \otimes \frac{1}{2} \sum_{k < l} \left[\sum_{\alpha} \Theta_{kl}(X, e_\alpha) e_\alpha \cdot \psi \right] &= \frac{1}{2} \sum_{k < l} (X \lrcorner \Theta_{kl}) \cdot \kappa_{r*}(f_k f_l) \cdot (\varphi \otimes \psi), \\ \frac{i}{2} \varphi \otimes \left[\sum_{\alpha} dA(X, e_\alpha) e_\alpha \cdot \psi \right] &= \frac{i}{2} \varphi \otimes X \lrcorner dA \cdot \psi. \end{aligned}$$

□

Proposition 3.5. *Let $\phi \in \Gamma(S)$. Then*

$$\sum_{i,j} e_i e_j \cdot R^{\theta,A}(e_i, e_j)(\phi) = \frac{R}{2} \phi + \sum_{k < l} \Theta_{kl} \cdot \kappa_{r*}(f_k f_l) \cdot \phi + i dA \cdot \phi,$$

where $\Theta_{kl} = \sum_{i < j} \Theta_{kl}(e_i, e_j) e_i \wedge e_j$ and R is the scalar curvature of M .

Proof. By (5),

$$\sum_{j=1}^n e_j \cdot R^{\theta,A}(e_i, e_j)(\phi) = -\frac{1}{2} \text{Ric}(e_i) \cdot \phi + \frac{1}{2} \sum_j \sum_{k < l} \Theta_{kl}(e_i, e_j) e_j \cdot \kappa_{r*}(f_k f_l) \cdot \phi + \frac{i}{2} e_i \lrcorner dA \cdot \phi.$$

Multiplying with e_i and summing over i , we get

$$\begin{aligned} \sum_{i,j} e_i e_j \cdot R^{\theta,A}(e_i, e_j)(\phi) &= -\frac{1}{2} \sum_i e_i \cdot \text{Ric}(e_i) \cdot \phi + \frac{1}{2} \sum_{k < l} \left[\sum_{i,j} \Theta_{kl}(e_i, e_j) e_i e_j \right] \cdot \kappa_{r*}(f_k f_l) \cdot \phi \\ &\quad + \frac{i}{2} \sum_i e_i \cdot e_i \lrcorner dA \cdot \phi. \end{aligned}$$

Now,

$$-\sum_i e_i \cdot \text{Ric}(e_i) = R,$$

where R denotes the scalar curvature of M . For k and l fixed,

$$\begin{aligned} \sum_{i,j} \Theta_{kl}(e_i, e_j) e_i e_j &= 2 \sum_{i < j} \Theta_{kl}(e_i, e_j) e_i e_j \\ &= 2 \Theta_{kl}, \\ \frac{i}{2} \sum_i e_i \cdot e_i \lrcorner dA \cdot \psi &= \frac{i}{2} \sum_{i,\alpha} dA(e_i, e_\alpha) e_i \cdot e_\alpha \cdot \psi \\ &= i \sum_{i < \alpha} dA(e_i, e_\alpha) e_i \cdot e_\alpha \cdot \psi \\ &= i dA \cdot \psi. \end{aligned}$$

□

Let us define

$$\begin{aligned} \Theta &= \sum_{k < l} \Theta_{kl} \otimes f_k f_l \in \wedge^2 T^* M \otimes \wedge^2 F, \\ \hat{\Theta} &= \sum_{k < l} \hat{\Theta}_{kl} \otimes f_k f_l \in \text{End}^-(TM) \otimes \wedge^2 F, \\ \eta^\phi &= \sum_{k < l} \eta_{kl}^\phi \otimes f_k f_l \in \wedge^2 T^* M \otimes \wedge^2 F, \\ \hat{\eta}^\phi &= \sum_{k < l} \hat{\eta}_{kl}^\phi \otimes f_k f_l \in \text{End}^-(TM) \otimes \wedge^2 F, \end{aligned}$$

where $\hat{\Theta}_{kl}$ denotes the skew-symmetric endomorphism associated to Θ_{kl} via the metric. Denote by

$$\tilde{\Theta} = (\mu_n \otimes \kappa_{r*})(\Theta),$$

the corresponding operator on twisted spinor fields. In order to simplify notation, we also define

$$\begin{aligned}\langle \Theta, \eta^\phi \rangle_0 &= \sum_{k < l} \sum_{i < j} \Theta_{kl}(e_i, e_j) \eta_{kl}^\phi(e_i, e_j), \\ \langle \hat{\Theta}, \hat{\eta}^\phi \rangle_1 &= \sum_{k < l} \text{tr}(\hat{\Theta}_{kl}(\hat{\eta}_{kl}^\phi)^T).\end{aligned}$$

Theorem 3.1 (Twisted Schrödinger-Lichnerowicz Formula). *Let $\phi \in \Gamma(S)$. Then*

$$\not{\partial}^{\theta, A}(\not{\partial}^{\theta, A}(\phi)) = \Delta^{\theta, A}(\phi) + \frac{R}{4}\phi + \frac{1}{2}\tilde{\Theta} \cdot \phi + \frac{i}{2}dA \cdot \phi \quad (6)$$

where R is the scalar curvature of the Riemannian manifold M .

Proof. Consider the difference

$$\not{\partial}^{\theta, A}(\not{\partial}^{\theta, A}(\phi)) - \Delta^{\theta, A}(\phi) = \sum_i \sum_{j \neq k} \langle \nabla_{e_i} e_j, e_k \rangle e_i e_k \cdot \nabla_{e_i}^{\theta, A} \phi + \sum_{i \neq j} e_i e_j \cdot \nabla_{e_j}^{\theta, A} \nabla_{e_i}^{\theta, A} \phi,$$

since

$$\sum_j \sum_{i=k} \langle \nabla_{e_i} e_j, e_k \rangle e_i e_k \nabla_{e_j}^{\theta, A} \phi = - \sum_j \text{div}(e_j) \nabla_{e_j}^{\theta, A} \phi.$$

Thus,

$$\begin{aligned}\not{\partial}^{\theta, A}(\not{\partial}^{\theta, A}(\phi)) - \Delta^{\theta, A}(\phi) &= \sum_j \sum_{i < k} \langle e_j, [e_k, e_i] \rangle e_i e_k \cdot \nabla_{e_i}^{\theta, A} \phi + \sum_{i < j} e_i e_j \cdot (\nabla_{e_i}^{\theta, A} \nabla_{e_j}^{\theta, A} - \nabla_{e_j}^{\theta, A} \nabla_{e_i}^{\theta, A}) \phi \\ &= \frac{1}{2} \sum_{i, j} e_i e_j R^{\theta, A}(e_i, e_j) \phi.\end{aligned}$$

The result follows from Proposition 3.5. \square

3.4 Bochner-type arguments

In this subsection we will prove some corollaries of the Schrödinger-Lichnerowicz-type formula and Bochner type arguments (cf. [10]). For the rest of the section, let us assume that the n -dimensional Riemannian $\text{Spin}^{c, r}$ manifold M is compact (without border) and connected.

3.4.1 Harmonic spinors

A twisted spinor field $\phi \in \Gamma(S)$ such that

$$\not{\partial}^{\theta, A} \phi = 0$$

will be called a *harmonic spinor*.

Corollary 3.2. *If $R \geq 2|\tilde{\Theta}| + 2|dA|$ everywhere (in pointwise operator norm), then a harmonic spinor is parallel. Furthermore, if the inequality is strict at a point, then there are no non-trivial harmonic spinors*

$$\ker(\not{\partial}^{\theta, A}) = \{0\}.$$

Proof. If $\phi \neq 0$ is a solution of

$$\not\partial^{\theta,A}(\phi) = 0,$$

by the twisted Schrödinger-Lichnerowicz formula (6)

$$0 = \Delta^{\theta,A}(\phi) + \frac{R}{4}\phi + \frac{1}{2}\tilde{\Theta} \cdot \phi + \frac{i}{2}dA \cdot \phi.$$

By taking hermitian product with ϕ and integrating over M we get

$$0 \geq \int_M |\nabla^{\theta,A}\phi|^2 + \frac{1}{4} \int_M (R - 2|\tilde{\Theta}| - 2|dA|) |\phi|^2.$$

Since

$$R - 2|\tilde{\Theta}| - 2|dA| \geq 0,$$

then

$$|\nabla^{\theta,A}\phi| = 0,$$

so that ϕ is parallel, has non-zero constant length and no zeroes.

Now, if

$$R - 2|\tilde{\Theta}| - 2|dA| > 0$$

at some point,

$$0 \geq |\phi|^2 \int_M (R - 2|\tilde{\Theta}| - 2|dA|) > 0.$$

□

Now notice that

$$\begin{aligned} \langle \tilde{\Theta} \cdot \phi, \phi \rangle &= \left\langle \sum_{k < l} \left[\sum_{i < j} \Theta_{kl}(e_i, e_j) e_i e_j \right] \cdot \kappa_{r*}(f_k f_l) \cdot \phi, \phi \right\rangle \\ &= \sum_{k < l} \sum_{i < j} \Theta_{kl}(e_i, e_j) \eta_{kl}^\phi(e_i, e_j) \\ &= \langle \Theta, \eta^\phi \rangle_0, \end{aligned}$$

which is a real number dependent on the curvature of the connection θ and the specific spinor ϕ .

Corollary 3.3. *If ϕ is such that*

$$R|\phi|^2 + 2\langle \Theta, \eta^\phi \rangle_0 + 2i\langle dA \cdot \phi, \phi \rangle \geq 0$$

everywhere, and the inequality is strict at a point, then

$$\not\partial^\theta(\phi) \neq 0.$$

Proof. Suppose $\phi \neq 0$ is such that

$$\not\partial^\theta(\phi) = 0.$$

Then, by (6)

$$0 = \int_M |\nabla^\theta \phi|^2 + \frac{1}{4} \int_M (R|\phi|^2 + 2\langle \Theta, \eta^\phi \rangle_0 + 2i\langle dA \cdot \phi, \phi \rangle) \geq 0,$$

so that ϕ is parallel, has non-zero constant length and no zeroes. Since

$$R|\phi|^2 + 2\langle \Theta, \eta^\phi \rangle_0 + 2i\langle dA \cdot \phi, \phi \rangle > 0$$

at some point,

$$0 \geq \int_M (R|\phi|^2 + 2\langle \Theta, \eta^\phi \rangle_0 + 2i\langle dA \cdot \phi, \phi \rangle) > 0.$$

□

3.4.2 Killing spinors

A twisted spinor field $\phi \in \Gamma(S)$ is called a *Killing spinor* if

$$\nabla_X^{\theta,A} \phi = \mu X \cdot \phi$$

for all $X \in \Gamma(TM)$, and μ a complex constant.

Corollary 3.4. *Suppose $\phi \neq 0$ is a Killing spinor with Killing constant μ . Then μ is either real or imaginary, and*

$$\mu^2 \geq \frac{1}{4n^2} \min_M (R - 2|\tilde{\Theta}| - 2|dA|).$$

If the inequality is attained, then ϕ is parallel, i.e. $\mu = 0$.

Proof. Recall that

$$\begin{aligned} \not{\partial}^{\theta,A}(\phi) &= \sum_{i=1}^n e_i \cdot \nabla_{e_i}^{\theta,A} \phi \\ &= -n\mu\phi. \end{aligned}$$

Then, by the twisted Schrödinger-Lichnerowicz formula (6)

$$n^2 \mu^2 \phi = \Delta^{\theta,A}(\phi) + \frac{R}{4} \phi + \frac{1}{2} \tilde{\Theta} \cdot \phi + \frac{i}{2} dA \cdot \phi.$$

By taking hermitian product with ϕ and integrating over M we get

$$\begin{aligned} n^2 \mu^2 \int_M |\phi|^2 &= \int_M |\nabla^{\theta,A} \phi|^2 + \int_M \frac{R}{4} |\phi|^2 + \int_M \frac{1}{2} \langle \tilde{\Theta} \cdot \phi, \phi \rangle + \frac{i}{2} \int_M \langle dA \cdot \phi, \phi \rangle \\ &\geq \frac{1}{4} \min_M (R - 2|\tilde{\Theta}| - 2|dA|) \int_M |\phi|^2, \end{aligned}$$

and the inequality follows. Since the right hand side of the equality above is a real number, μ must be either real or imaginary. Now, if the inequality is attained,

$$\int_M |\nabla^{\theta,A} \phi|^2 = 0 \quad \text{and} \quad \nabla^{\theta,A} \phi = 0.$$

□

Corollary 3.5. *Suppose $\phi \in \Gamma(S)$ is a Dirac eigenspinor*

$$\not{\partial}^{\theta,A} \phi = \lambda \phi.$$

Then

$$\lambda^2 \geq \frac{n}{4(n-1)} \left(\min_M (R - 2|\tilde{\Theta}| - 2|dA|) \right).$$

If the lower bound is non-negative and is attained, the spinor ϕ is a real Killing spinor with Killing constant

$$\mu = \pm \frac{1}{2} \sqrt{\frac{1}{n(n-1)} \min_M (R - 2|\tilde{\Theta}| - 2|dA|)}.$$

Proof. Let $h : M \rightarrow \mathbb{R}$ be a fixed smooth function. Consider the following metric connection on the twisted Spin bundle

$$\nabla_X^h \phi = \nabla_X^{\theta, A} \phi + hX \cdot \phi.$$

Let

$$\Delta^h(\phi) = - \sum_{i=1}^n \nabla_{e_i}^h \nabla_{e_i}^h \phi - \sum_{i=1}^n \operatorname{div}(e_i) \nabla_{e_i}^h \phi,$$

be the Laplacian for this connection and recall that

$$|\nabla^h \phi|^2 = \sum_{i=1}^n |\nabla_{e_i}^{\theta, A} \phi + h e_i \cdot \phi|^2.$$

Then, by (6)

$$\begin{aligned} (\not{\partial}^{\theta, A} - h) \circ (\not{\partial}^{\theta, A} - h)(\phi) &= \not{\partial}^{\theta, A}(\not{\partial}^{\theta, A} \phi) - 2h \not{\partial}^{\theta, A} \phi - \operatorname{grad}(h) \cdot \phi + h^2 \phi \\ &= \Delta^{\theta, A}(\phi) + \frac{R}{4} \phi + \frac{1}{2} \tilde{\Theta} \cdot \phi + \frac{i}{2} dA \cdot \phi - 2h \not{\partial}^{\theta, A} \phi - \operatorname{grad}(h) \cdot \phi + h^2 \phi. \end{aligned}$$

On the other hand,

$$\Delta^h \phi = \Delta^{\theta, A} \phi - 2h \not{\partial}^{\theta, A} \phi - \operatorname{grad}(h) \cdot \phi + nh^2 \phi.$$

Thus

$$(\not{\partial}^{\theta, A} - h) \circ (\not{\partial}^{\theta, A} - h)(\phi) = \Delta^h(\phi) + \frac{R}{4} \phi + \frac{1}{2} \tilde{\Theta} \cdot \phi + \frac{i}{2} dA \cdot \phi + (1 - n)h^2 \phi$$

By using $\not{\partial}^{\theta, A} \phi = \lambda \phi$, setting $h = \frac{\lambda}{n}$, taking hermitian product with ϕ and integrating over M we get

$$\lambda^2 \left(\frac{n-1}{n} \right)^2 \int_M |\phi|^2 = \int_M |\nabla^{\lambda/n} \phi|^2 + \lambda^2 \frac{1-n}{n^2} \int_M |\phi|^2 + \int_M \frac{R}{4} |\phi|^2 + \int_M \frac{1}{2} \langle \tilde{\Theta} \cdot \phi, \phi \rangle + \frac{i}{2} \int_M \langle dA \cdot \phi, \phi \rangle$$

so that

$$\lambda^2 \left(\frac{n-1}{n} \right) \int_M |\phi|^2 \geq \frac{1}{4} \min(R - 2|\tilde{\Theta}| - 2|dA|) \int_M |\phi|^2.$$

If the lower bound is attained,

$$\int_M |\nabla^{\lambda/n} \phi|^2 = 0,$$

i.e.

$$\nabla^{\lambda/n} \phi = 0.$$

□

4 CR structures of arbitrary codimension

In this section we will explore the twisted spinorial geometry associated to almost CR structures. We carry out the spinorial characterization and explore some integrability conditions of almost CR structures implied by assuming the typical conditions on spinors, such as being parallel or Killing, but just in prescribed directions.

4.1 Spinorial characterization of almost CR (hermitian) structures

Definition 4.1. Let M be a smooth $(2m + r)$ -dimensional smooth manifold.

- An almost CR structure on a manifold M consists of a sub-bundle $D \subset TM$ and a bundle automorphism J of D such that $J^2 = -\text{Id}_D$.
- An almost CR hermitian structure on M is an almost CR structure whose almost complex structure is orthogonal with respect to the metric.

Remark. Given an almost CR structure on M we can introduce an (auxiliary) compatible metric as follows. Take any Riemannian metric g_0 on M and consider the orthogonal complement D^\perp of D with respect to this metric. Let g_1 and g_2 denote the restrictions of g_0 to D and D^\perp respectively. Average g_1 with respect to J and call it g_3 . Finally, consider the metric $g = g_3 \oplus g_2$.

Definition 4.2. Let M be an oriented Riemannian $\text{Spin}^{c,r}(n)$ manifold and S the associated twisted spinor bundle. A (nowhere zero) spinor field $\phi \in \Gamma(S)$ is called partially pure if $\phi_x \in S_x$ is partially pure at each point $x \in M$.

Theorem 4.1. Let M be an oriented n -dimensional Riemannian manifold. Then the following two statements are equivalent:

- M admits a twisted $\text{Spin}^{c,r}$ structure carrying a partially pure spinor field $\phi \in \Gamma(S)$, where S denotes the associated twisted spinor bundle.
- M admits an almost CR hermitian structure of codimension r .

Proof. If the manifold M admits a partially pure spinor field $\phi \in \Gamma(S)$, the subspaces V_x^ϕ determine a smooth distribution of even rank $n - r$ carrying an almost complex structure.

Conversely, if M has an orthogonal almost CR hermitian structure of codimension r , the tangent bundle decomposes orthogonally as

$$TM = D \oplus D^\perp,$$

where D has real rank $2m = n - r$ and admits an almost complex structure, and D^\perp is the oriented orthogonal complement. The structure group of the Riemannian manifold M reduces to $U(m) \times SO(r)$ and, by Lemma 2.1, there is a monomorphism

$$U(m) \times SO(r) \hookrightarrow \text{Spin}^{c,r}(2m + r)$$

with image $\widehat{U(m) \times SO(r)}$, which allows us to associate a $\text{Spin}^{c,r}(n)$ principal bundle P on M , i.e. a $\text{Spin}^{c,r}$ structure. Note that the corresponding twisted spinor bundle S decomposes under $\widehat{U(m) \times SO(r)}$ as follows

$$\begin{aligned} S &= \left[\kappa_D^{-1/2} \otimes \Sigma(D^\perp) \right] \otimes \Delta(M) \\ &= \left[\kappa_D^{-1/2} \otimes \Sigma(D^\perp) \right] \otimes \Delta(D^\perp) \otimes \Delta(D) \\ &= \left[\kappa_D^{-1/2} \otimes \Sigma(D^\perp) \right] \otimes \Delta(D^\perp) \otimes \left[(\wedge^* D^{0,1}) \otimes \kappa_D^{1/2} \right] \\ &= \left[\Sigma(D^\perp) \otimes \Delta(D^\perp) \right] \otimes \left[\wedge^* D^{0,1} \right], \end{aligned}$$

where $\kappa_D = \wedge^m D^{1,0}$. We see that it contains a rank 1 trivial subbundle generated by the partially pure spinor given in (1) with stabilizer $\widehat{U(m) \times SO(r)}$, i.e. M admits a global partially pure spinor field. \square

Example. Recall from Subsection 2.6 that

$$T_{\text{Id}}\mathcal{G}_{m,1,r} \cong [[\wedge^2 \mathbb{C}^m]] \oplus [[\mathbb{C}^m]] \otimes \mathbb{R}^r \oplus [[\mathbb{C}^m]] \oplus \mathbb{R}^r$$

For the sake of clarity, consider $m = 2$, $r = 2$ and $\mathbb{R}^7 = \mathbb{R}^4 \oplus \mathbb{R}^2 \oplus \mathbb{R}^1$, where the first summand \mathbb{R}^4 is endowed with the standard complex structure

$$\begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & 1 & 0 \end{pmatrix}.$$

The different summands in the decomposition

$$T_{\text{Id}}\mathcal{G} \cong [[\wedge^2 \mathbb{C}^2]] \oplus [[\mathbb{C}^2]] \otimes \mathbb{R}^2 \oplus [[\mathbb{C}^2]] \oplus \mathbb{R}^2.$$

correspond to skew-symmetric matrices as follows:

$$\begin{aligned} [[\wedge^2 \mathbb{C}^2]] &= \left\{ \begin{pmatrix} 0 & 0 & b_1 & b_2 & & & \\ & 0 & b_2 & -b_1 & & & \\ -b_1 & -b_2 & 0 & & & & \\ -b_2 & b_1 & & 0 & & & \\ & & & & 0 & & \\ & & & & & 0 & \\ & & & & & & 0 \end{pmatrix} : b_1, b_2 \in \mathbb{R} \right\}, \\ [[\mathbb{C}^2]] \otimes \mathbb{R}^2 &= \left\{ \begin{pmatrix} 0 & & & & c_1 & c_2 & \\ & 0 & & & c_3 & c_4 & \\ & & 0 & & c_5 & c_6 & \\ & & & 0 & c_7 & c_8 & \\ -c_1 & -c_3 & -c_5 & -c_7 & 0 & & \\ -c_2 & -c_4 & -c_6 & -c_8 & & 0 & \\ & & & & & & 0 \end{pmatrix} : c_j \in \mathbb{R}, j = 1, \dots, 8 \right\}, \\ [[\mathbb{C}^2]] &= \left\{ \begin{pmatrix} 0 & & & & d_1 & \\ & 0 & & & d_2 & \\ & & 0 & & d_3 & \\ & & & 0 & d_4 & \\ & & & & 0 & \\ -d_1 & -d_2 & -d_3 & -d_4 & & 0 \end{pmatrix} : d_j \in \mathbb{R}, j = 1, \dots, 4 \right\}, \\ \mathbb{R}^2 &= \left\{ \begin{pmatrix} 0 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & \delta_1 & \\ & & & & 0 & \delta_2 \\ & & -\delta_1 & -\delta_2 & 0 & \end{pmatrix} : \delta_1, \delta_2 \in \mathbb{R} \right\}. \end{aligned}$$

The induced complex structure on $[[\wedge^2 \mathbb{C}^2]] \oplus [[\mathbb{C}^2]] \otimes \mathbb{R}^2 \oplus [[\mathbb{C}^2]]$, which respects each summand, is

$$J \begin{pmatrix} 0 & 0 & b_1 & b_2 & c_1 & c_2 & d_1 \\ & 0 & b_2 & -b_1 & c_3 & c_4 & d_2 \\ & & 0 & 0 & c_5 & c_6 & d_3 \\ & & & 0 & c_7 & c_8 & d_4 \\ & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -b_2 & b_1 & -c_3 & -c_4 & -d_2 \\ & 0 & b_1 & b_2 & c_1 & c_2 & d_1 \\ & & 0 & 0 & -c_7 & -c_8 & -d_4 \\ & & & 0 & c_5 & c_6 & d_3 \\ & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{pmatrix}$$

where we have only written the upper triangle part for notational simplicity.

Thus, this example gives us several candidates of distributions carrying an almost complex structure, as well as their orthogonal complements (with respect to the natural metric):

$$\begin{aligned}
\begin{cases} D_1 &= [[\wedge^2 \mathbb{C}^m]] \\ D_1^\perp &= [[\mathbb{C}^m]] \otimes \mathbb{R}^r \oplus [[\mathbb{C}^m]] \oplus \mathbb{R}^r \end{cases} \\
\begin{cases} D_2 &= [[\mathbb{C}^m]] \otimes \mathbb{R}^r \\ D_2^\perp &= [[\wedge^2 \mathbb{C}^m]] \oplus [[\mathbb{C}^m]] \oplus \mathbb{R}^r \end{cases} \\
\begin{cases} D_3 &= [[\mathbb{C}^m]] \\ D_3^\perp &= [[\wedge^2 \mathbb{C}^m]] \oplus [[\mathbb{C}^m]] \otimes \mathbb{R}^r \oplus \mathbb{R}^r \end{cases} \\
\begin{cases} D_4 &= [[\wedge^2 \mathbb{C}^m]] \oplus [[\mathbb{C}^m]] \otimes \mathbb{R}^r \\ D_4^\perp &= [[\mathbb{C}^m]] \oplus \mathbb{R}^r \end{cases} \\
\begin{cases} D_5 &= [[\wedge^2 \mathbb{C}^m]] \oplus [[\mathbb{C}^m]] \\ D_5^\perp &= [[\mathbb{C}^m]] \otimes \mathbb{R}^r \oplus \mathbb{R}^r \end{cases} \\
\begin{cases} D_6 &= [[\mathbb{C}^m]] \otimes \mathbb{R}^r \oplus [[\mathbb{C}^m]] \\ D_6^\perp &= [[\wedge^2 \mathbb{C}^m]] \oplus \mathbb{R}^r \end{cases} \\
\begin{cases} D_7 &= [[\wedge^2 \mathbb{C}^m]] \oplus [[\mathbb{C}^m]] \otimes \mathbb{R}^r \oplus [[\mathbb{C}^m]] \\ D_7^\perp &= \mathbb{R}^r \end{cases}
\end{aligned}$$

By computing the Lie brackets at the Lie algebra level, we see that the distributions D_1, D_4, D_6^\perp and D_7^\perp are involutive with their foliations corresponding to the fibers of the following four fibrations

$$\begin{aligned}
\frac{SO(2m)}{U(m)} &\hookrightarrow \frac{SO(2m+r+1)}{U(m) \times SO(r)} \\
&\downarrow \\
&\frac{SO(2m+r+1)}{SO(2m) \times SO(r)}, \\
\frac{SO(2m+r)}{U(m) \times SO(r)} &\hookrightarrow \frac{SO(2m+r+1)}{U(m) \times SO(r)} \\
&\downarrow \\
&S^{2m+r}, \\
\frac{SO(2m)}{U(m)} \times S^r &\hookrightarrow \frac{SO(2m+r+1)}{U(m) \times SO(r)} \\
&\downarrow \\
&\frac{SO(2m+r+1)}{SO(2m) \times SO(r+1)}, \\
S^r &\hookrightarrow \frac{SO(2m+r+1)}{U(m) \times SO(r)} \\
&\downarrow \\
&\frac{SO(2m+r+1)}{U(m) \times SO(r+1)},
\end{aligned}$$

respectively.

4.2 Adapted connection for almost CR-hermitian manifolds

Before we proceed with the characterizations of integrability conditions, we need to give (at least) a choice of connection on the relevant bundles of an almost CR hermitian manifold, or equivalently, on a $\text{Spin}^{c,r}$ manifold carrying a partially pure spinor.

As we mentioned earlier, we can adapt a metric on an almost CR manifold M in order to make it an almost CR hermitian manifold. Let us fix one such metric and its Levi-Civita connection 1-form

ω and covariant derivative ∇ . The metric determines the orthogonal complement D^\perp and gives us a covariant derivative as follows

$$\begin{aligned}\nabla_X^{D^\perp} : \Gamma(D^\perp) &\longrightarrow \Gamma(D^\perp) \\ W &\mapsto \text{proj}_{D^\perp}(\nabla_X W)\end{aligned}$$

for $X \in \Gamma(TM)$, whose local connection 1-forms and curvature 2-forms will be denoted by $\theta_{kl}^{D^\perp}$ and $\Theta_{kl}^{D^\perp}$ respectively, $1 \leq k < l \leq r$. The analogous connection on D is given by the covariant derivative

$$\begin{aligned}\nabla_X^D : \Gamma(D) &\longrightarrow \Gamma(D) \\ W &\mapsto \text{proj}_D(\nabla_X W)\end{aligned}$$

for $X \in \Gamma(TM)$. However, we need to induce a connection on κ_D^{-1} . Thus, we consider the hermitian connection for (D, \langle, \rangle, J) defined by

$$\tilde{\nabla}_X^D Y = \nabla_X^D Y + \frac{1}{2}(\nabla_X^D J)(JY).$$

so that

$$\tilde{\nabla}^D J = 0.$$

$\tilde{\nabla}^D$ induces a covariant derivative $\tilde{\nabla}^{\kappa_D^{-1}}$ on the anticanonical bundle κ_D^{-1} of D , whose local connection 1-form will be denoted by $i\tilde{A}^D$. More precisely, if (e_1, \dots, e_n) is a local orthonormal frame of TM such that

$$\begin{aligned}D &= \text{span}(e_1, \dots, e_{2m}), \\ e_{2s} &= J(e_{2s-1}), \\ D^\perp &= \text{span}(e_{2m+1}, \dots, e_{2m+r}),\end{aligned}$$

for $1 \leq s \leq m$ and $1 \leq k < l \leq r$. and the matrix of connection 1-forms of $\tilde{\nabla}^D$ is

$$\begin{pmatrix} 0 & \tilde{\omega}_{1,2} & & \tilde{\omega}_{1,2m-1} & \tilde{\omega}_{1,2m} \\ -\tilde{\omega}_{12} & 0 & & -\tilde{\omega}_{1,2m} & \tilde{\omega}_{1,2m-1} \\ & & \ddots & & \\ -\tilde{\omega}_{1,2m-1} & \tilde{\omega}_{1,2m} & & 0 & \tilde{\omega}_{2m-1,2m} \\ -\tilde{\omega}_{1,2m} & -\tilde{\omega}_{1,2m-1} & & -\tilde{\omega}_{2m-1,2m} & 0 \end{pmatrix},$$

the induced connection on $\kappa_D^{-1} = \bigwedge^m D^{0,1}$ is

$$i\tilde{A} = -i[\tilde{\omega}_{1,2} + \dots + \tilde{\omega}_{2m-1,2m}].$$

By using ∇ , ∇^{D^\perp} and the unitary connection $i\tilde{A}^D$, we can define a connection ∇^S on the globally defined twisted spinor vector bundle $S = [\kappa_D^{-1/2} \otimes \Sigma(D^\perp)] \otimes \Delta(M)$ which is compatible with Clifford multiplication.

4.3 Spinorial characterization of integrability

Definition 4.3. Let M be a smooth $2m + r$ dimensional smooth manifold. An almost CR structure is called a CR structure if for every $X, Y \in \Gamma(D)$

- $[X, Y] - [J(X), J(Y)] \in \Gamma(D)$,
- $[J(X), Y] + [X, J(Y)] \in \Gamma(D)$,

- $J([X, Y] - [J(X), J(Y)]) = [J(X), Y] + [X, J(Y)]$.

Example. By computing the relevant combinations of brackets one can check that the distributions D_1, D_4, D_5 and D_7 on $\mathcal{G}_{m,1,r}$ are CR-integrable.

Theorem 4.2. *Let M be an oriented n -dimensional Riemannian manifold. The following are equivalent:*

- (i) M is endowed with a CR hermitian structure of codimension r .
- (ii) M admits a twisted $\text{Spin}^{c,r}(n)$ structure and a twisted spinor bundle S carrying a partially pure spinor field $\phi \in \Gamma(S)$ which satisfies

$$(X - iJ^\phi(X)) \cdot \nabla_{(Y - iJ^\phi(Y))}^S \phi = (Y - iJ^\phi(Y)) \cdot \nabla_{(X - iJ^\phi(X))}^S \phi,$$

for every $X, Y \in \Gamma(V^\phi)$, where ∇^S is the covariant derivative described in subsection 4.2.

Proof. First, let us assume (i), i.e. M admits a CR hermitian structure. By Theorem 4.1, M admits a twisted spinor vector bundle $S = [\kappa_D^{-1/2} \otimes \Sigma(D^\perp)] \otimes \Delta(M)$ carrying a partially pure spinor field $\phi \in \Gamma(S)$ such that $V^\phi = D$, $J^\phi = J$ and

$$(X - iJX) \cdot \phi = 0$$

for every $X \in \Gamma(D)$. By differentiating this identity

$$(\nabla_Y X - i\nabla_Y(JX)) \cdot \phi + (X - iJX) \cdot \nabla_Y^S \phi = 0, \quad (7)$$

and similarly

$$(\nabla_X Y - i\nabla_X(JY)) \cdot \phi + (Y - iJY) \cdot \nabla_X^S \phi = 0, \quad (8)$$

By subtracting (7) from (8)

$$([X, Y] - i\nabla_X(JY) + i\nabla_Y(JX)) \cdot \phi = (X - iJX) \cdot \nabla_Y^S \phi - (Y - iJY) \cdot \nabla_X^S \phi, \quad (9)$$

By substituting X with JX , and Y with JY in (9)

$$([JX, JY] + i\nabla_{JX}(Y) - i\nabla_{JY}(X)) \cdot \phi = -(X - iJX) \cdot \nabla_{-iJY}^S \phi + (Y - iJY) \cdot \nabla_{-iJX}^S \phi, \quad (10)$$

Subtract (10) from (9)

$$([X, Y] - [JX, JY] - i([X, JY] + [JX, Y])) \cdot \phi = (X - iJX) \cdot \nabla_{-iJY}^S \phi - (Y - iJY) \cdot \nabla_{-iJX}^S \phi. \quad (11)$$

Since $[X, Y] - [JX, JY] \in \Gamma(D)$

$$\begin{aligned} ([X, Y] - [JX, JY]) \cdot \phi &= iJ([X, Y] - [JX, JY]) \cdot \phi \\ &= i([J(X), Y] + [X, J(Y)]) \cdot \phi, \end{aligned}$$

so that the left hand side of (11) vanishes.

Conversely, let us assume (ii). Then, the subbundle V^ϕ together with its endomorphism J^ϕ provide an almost CR hermitian structure on M . By considering the equation

$$(X - iJ^\phi X) \cdot \phi = 0$$

for all $X \in V^\phi$, and performing the same calculations as before, we arrive at

$$\begin{aligned} ([X, Y] - [J^\phi X, J^\phi Y] - i([X, J^\phi Y] + [J^\phi X, Y])) \cdot \phi &= (X - iJ^\phi X) \cdot \nabla_{-iJ^\phi Y}^S \phi - (Y - iJ^\phi Y) \cdot \nabla_{-iJ^\phi X}^S \phi \\ &= 0, \end{aligned}$$

i.e.

$$([X, Y] - [J^\phi X, J^\phi Y]) \cdot \phi = i([X, J^\phi Y] + [J^\phi X, Y]) \cdot \phi,$$

which implies

- $[X, Y] - [J^\phi(X), J^\phi(Y)] \in \Gamma(V^\phi)$,
- $[J^\phi(X), Y] + [X, J^\phi(Y)] \in \Gamma(V^\phi)$,
- $J^\phi([X, Y] - [J^\phi(X), J^\phi(Y)]) = [J^\phi(X), Y] + [X, J^\phi(Y)]$,

since ϕ is a partially pure spinor. □

4.4 D -parallel partially pure spinor

The following theorem is motivated by the condition

$$\nabla_X^S \phi = 0$$

for all $X \in \Gamma(D)$, i.e. ϕ being D -parallel.

Theorem 4.3. *Let M be an oriented n -dimensional Riemannian manifold. The following are equivalent:*

- (i) *M admits a twisted $\text{Spin}^{c,r}(n)$ structure and a twisted spinor bundle S carrying a partially pure spinor field $\phi \in \Gamma(S)$ satisfying*

$$(Y - iJ^\phi(Y)) \cdot \nabla_X^S \phi = 0$$

for every $X, Y \in \Gamma(V^\phi)$, where ∇^S is the covariant derivative described in subsection 4.2.

- (ii) *M is endowed with an almost CR hermitian structure of codimension r , where D and J are D -parallel. (In particular, J restricts to a Kähler structure on each leaf of the integral foliation of D , and D^\perp is D -parallel.)*

Proof. Let us assume (i) and $D = V^\phi$, $D^\perp = (V^\phi)^\perp$, $J = J^\phi$. Since

$$(Y - iJ^\phi Y) \cdot \phi = 0$$

for every $Y \in \Gamma(V^\phi)$, if $X \in \Gamma(V^\phi)$

$$\begin{aligned} 0 &= \nabla_X^S((Y - iJ^\phi Y) \cdot \phi) \\ &= (\nabla_X Y - i\nabla_X(J^\phi Y)) \cdot \phi + (Y - iJ^\phi Y) \cdot \nabla_X^S \phi \\ &= (\nabla_X Y - i\nabla_X(J^\phi Y)) \cdot \phi, \end{aligned}$$

which means

$$\begin{aligned} \nabla_X Y &\in D, \\ \nabla_X(JY) &= J(\nabla_X Y). \end{aligned}$$

i.e. D and J are D -parallel so that the leaves of this totally geodesic foliation are Kähler manifolds. If $u \in \Gamma(D^\perp)$

$$\langle Y, u \rangle = 0$$

for every $Y \in \Gamma(D)$, so that for every $X \in \Gamma(D)$

$$\begin{aligned} 0 &= X \langle Y, u \rangle \\ &= \langle Y, \nabla_X u \rangle \end{aligned}$$

since D is D -parallel, thus showing that $\nabla_X u \in \Gamma(D^\perp)$.

Conversely, if M admits an almost CR hermitian structure. By Theorem 4.1, M admits a twisted Spin structure and a twisted spinor bundle S endowed with a connection ∇^S , carrying a partially pure spinor field $\phi \in \Gamma(S)$ such that $V^\phi = D$, $J^\phi = J$ and

$$(Y - iJY) \cdot \phi = 0$$

for every $Y \in \Gamma(D)$. Thus, for $X \in \Gamma(D)$,

$$\begin{aligned} 0 &= \nabla_X((Y - iJY) \cdot \phi) \\ &= (\nabla_X Y - iJ(\nabla_X Y)) \cdot \phi + (Y - iJ(Y)) \cdot \nabla_X^S \phi \\ &= (Y - iJ(Y)) \cdot \nabla_X^S \phi \end{aligned}$$

since J is D -parallel. As before, D^\perp is D -parallel. \square

Example. The space $\mathcal{G}_{m,1,r}$ admits the CR distribution D_1 satisfying the hypotheses of Theorem 4.3, as can be seen from the fibration:

$$\begin{array}{ccc} \frac{SO(2m)}{U(m)} & \hookrightarrow & \frac{SO(2m+r+1)}{U(m) \times SO(r)} \\ & & \downarrow \\ & & \frac{SO(2m+r+1)}{SO(2m) \times SO(r)}. \end{array}$$

When the partially pure spinor is parallel, we can actually say more about the foliation leaves' Ricci curvature.

Theorem 4.4. *Let M be a $Spin^{c,r}$ n -dimensional Riemannian manifold such that its twisted spinor bundle S admits a partially pure spinor field $\phi \in \Gamma(S)$ satisfying*

$$\nabla_X^S \phi = 0$$

for every $X \in \Gamma(V^\phi)$, where ∇^S is the covariant derivative described in subsection 4.2. Then

1. *The Ricci tensor of V^ϕ satisfies*

$$\text{Ric}^{V^\phi} = \left[\text{proj}_{V^\phi} \circ \widehat{dA}|_{V^\phi} \right] \circ J^\phi, \quad (12)$$

where \widehat{dA} denotes the skew-symmetric endomorphism determined by dA (the curvature of the connection 1-form on the auxiliary principal $U(1)$ bundle) and metric dualization.

2. *The scalar curvature is given by*

$$R^{V^\phi} = \text{tr} \left(\left[\text{proj}_{V^\phi} \circ \widehat{dA}|_{V^\phi} \right] \circ J^\phi \right).$$

3. *If the connection A on the auxiliary bundle L is flat along an integral leaf of V^ϕ , then the leaf is Calabi-Yau.*

Remark. The identity (12) tells us that $\text{proj}_{V^\phi} \circ \widehat{dA}|_{V^\phi}$, restricted to the leaves of the corresponding foliation, equals their Ricci form.

Proof. Since ϕ is partially pure, $n = 2m + r$ where $\text{rank}(V^\phi) = 2m$ and $\text{rank}((V^\phi)^\perp) = r$. Let (e_1, \dots, e_n) and (f_1, \dots, f_r) be local orthonormal frames of TM and F respectively, such that

$$\begin{aligned} V^\phi &= \text{span}(e_1, \dots, e_{2m}), \\ e_{2j} &= J^\phi(e_{2j-1}), \end{aligned}$$

$$\begin{aligned}(V^\phi)^\perp &= \text{span}(e_{2m+1}, \dots, e_{2m+r}), \\ \eta_{kl}^\phi &= e_{2m+k} \wedge e_{2m+l},\end{aligned}$$

for $1 \leq j \leq m$ and $1 \leq k < l \leq r$. If $X \in \Gamma(V^\phi)$ and $1 \leq \alpha \leq 2m$ then, by Theorem 4.3, $[X, e_\alpha] \in \Gamma(V^\phi)$ and

$$\begin{aligned}R^M(X, e_\alpha)e_i &\in \Gamma(V^\phi) & \text{if } 1 \leq i \leq 2m, \\ R^M(X, e_\alpha)e_i &\in \Gamma((V^\phi)^\perp) & \text{if } 2m+1 \leq i \leq 2m+r.\end{aligned}$$

so that

$$\begin{aligned}\langle R^M(X, e_\alpha)e_i, e_j \rangle &= 0 & \text{if } 1 \leq i \leq 2m, 2m+1 \leq j \leq 2m+r, \\ \langle R^M(X, e_\alpha)e_i, e_j \rangle &= 0 & \text{if } 2m+1 \leq i \leq 2m+r, 1 \leq j \leq 2m.\end{aligned}$$

For ϕ ,

$$\begin{aligned}0 &= R^{\theta, A}(X, e_\alpha)\phi \\ &= \frac{1}{2} \sum_{1 \leq i < j \leq n} \langle R^M(X, e_\alpha)e_i, e_j \rangle e_i e_j \cdot \phi + \frac{1}{2} \sum_{1 \leq k < l \leq r} \Theta_{kl}(X, e_\alpha) \kappa_{r*}(f_k f_l) \cdot \phi + \frac{i}{2} dA(X, e_\alpha)\phi \\ &= \frac{1}{2} \sum_{1 \leq i < j \leq 2m} \langle R^M(X, e_\alpha)e_i, e_j \rangle e_i e_j \cdot \phi + \frac{1}{2} \sum_{1 \leq k < l \leq r} \langle R^M(X, e_\alpha)e_{2m+k}, e_{2m+l} \rangle e_{2m+k} e_{2m+l} \cdot \phi \\ &\quad + \frac{1}{2} \sum_{1 \leq k < l \leq r} \Theta_{kl}(X, e_\alpha) \kappa_{r*}(f_k f_l) \cdot \phi + \frac{i}{2} dA(X, e_\alpha)\phi \\ &= \frac{1}{2} \sum_{1 \leq i < j \leq 2m} \langle R^M(X, e_\alpha)e_i, e_j \rangle e_i e_j \cdot \phi + \frac{1}{2} \sum_{1 \leq k < l \leq r} \langle R^M(X, e_\alpha)e_{2m+k}, e_{2m+l} \rangle \kappa_{r*}(f_k f_l) \cdot \phi \\ &\quad + \frac{1}{2} \sum_{1 \leq k < l \leq r} \Theta_{kl}(X, e_\alpha) \kappa_{r*}(f_k f_l) \cdot \phi + \frac{i}{2} dA(X, e_\alpha)\phi,\end{aligned}$$

where Θ_{kl} denote the local curvature 2-forms of the auxiliary connection on $PSO(r)$. Multiply by e_α and sum over α , $1 \leq \alpha \leq 2m$,

$$\begin{aligned}0 &= \sum_{\alpha=1}^{2m} \sum_{1 \leq i < j \leq 2m} \langle R^M(X, e_\alpha)e_i, e_j \rangle e_\alpha e_i e_j \cdot \phi + \sum_{\alpha=1}^{2m} \sum_{1 \leq k < l \leq r} \langle R^M(X, e_\alpha)e_{2m+k}, e_{2m+l} \rangle e_\alpha \cdot \kappa_{r*}(f_k f_l) \cdot \phi \\ &\quad + \sum_{\alpha=1}^{2m} \sum_{1 \leq k < l \leq r} \Theta_{kl}(X, e_\alpha) e_\alpha \cdot \kappa_{r*}(f_k f_l) \cdot \phi + i \sum_{\alpha=1}^{2m} dA(X, e_\alpha) e_\alpha \cdot \phi \\ &= -\text{Ric}^{V^\phi}(X) \cdot \phi + \sum_{\alpha=1}^{2m} \sum_{1 \leq k < l \leq r} \langle R^M(X, e_\alpha)e_{2m+k}, e_{2m+l} \rangle e_\alpha \cdot \kappa_{r*}(f_k f_l) \cdot \phi \\ &\quad + \sum_{\alpha=1}^{2m} \sum_{1 \leq k < l \leq r} \Theta_{kl}(X, e_\alpha) e_\alpha \cdot \kappa_{r*}(f_k f_l) \cdot \phi + i \sum_{\alpha=1}^{2m} dA(X, e_\alpha) e_\alpha \cdot \phi.\end{aligned}$$

By taking the real part of the hermitian inner product with $e_i \cdot \phi$, $1 \leq i \leq 2m$,

$$\begin{aligned}\text{Re} \langle \text{Ric}^{V^\phi}(e_j) \cdot \phi, e_i \cdot \phi \rangle &= \langle \text{Ric}^{V^\phi}(e_j), e_i \rangle |\phi|^2 \\ &= \text{Ric}_{ij}^{V^\phi},\end{aligned}$$

since $|\phi| = 1$, where now $1 \leq j \leq 2m$. On the other hand,

$$\text{Re} \langle \text{Ric}^{V^\phi}(e_j) \cdot \phi, e_i \cdot \phi \rangle = \text{Re} \left\langle \sum_{\alpha=1}^{2m} \sum_{1 \leq k < l \leq r} \langle R^M(e_j, e_\alpha)e_{2m+k}, e_{2m+l} \rangle e_\alpha \cdot \kappa_{r*}(f_k f_l) \cdot \phi, e_i \cdot \phi \right\rangle$$

$$\begin{aligned}
& + \operatorname{Re} \left\langle \sum_{\alpha=1}^{2m} \sum_{1 \leq k < l \leq r} \Theta_{kl}(e_j, e_\alpha) e_\alpha \cdot \kappa_{r*}(f_k f_l) \cdot \phi, e_i \cdot \phi \right\rangle \\
& + \operatorname{Re} \left\langle i \sum_{\alpha=1}^{2m} dA(e_j, e_\alpha) e_\alpha \cdot \phi, e_i \cdot \phi \right\rangle \\
& = \sum_{\alpha=1}^{2m} dA(e_j, e_\alpha) \operatorname{Re} \langle i e_\alpha \cdot \phi, e_i \cdot \phi \rangle \\
& = - \sum_{\alpha=1}^{2m} dA(e_j, e_\alpha) \operatorname{Re} \langle J^\phi(e_\alpha) \cdot \phi, e_i \cdot \phi \rangle \\
& = \sum_{\alpha=1}^{2m} dA(e_j, e_\alpha) \langle J^\phi(e_\alpha), e_i \rangle |\phi|^2 \\
& = \sum_{\alpha=1}^{2m} dA(e_j, e_\alpha) J_{i\alpha}^\phi.
\end{aligned}$$

Thus

$$\operatorname{Ric}^{V^\phi} = \left[\operatorname{proj}_{V^\phi} \circ \widehat{dA}|_{V^\phi} \right] \circ J^\phi,$$

where \widehat{dA} denotes the skew-symmetric endomorphism determined by dA and metric dualization. \square

Remark. On each Kähler leaf, the spinor ϕ restricts to a parallel pure Spin^c spinor field.

4.5 D^\perp -parallel partially pure spinor

The following theorem is motivated by the condition

$$\nabla_u^S \phi = \lambda u \cdot \phi$$

for all $u \in \Gamma(D^\perp)$, $\lambda \in \mathbb{R}$, i.e. ϕ being a real D^\perp -Killing spinor.

Theorem 4.5. *Let M be an oriented n -dimensional Riemannian manifold. The following are equivalent:*

- (i) *M admits a twisted $\operatorname{Spin}^{c,r}(n)$ structure and a twisted spinor bundle S carrying a partially pure spinor field $\phi \in \Gamma(S)$ satisfying*

$$(Y - iJ^\phi(Y)) \cdot \nabla_u^S \phi = 0$$

for every $Y \in \Gamma(V^\phi)$ and $u \in \Gamma((V^\phi)^\perp)$, where ∇^S is the covariant derivative described in subsection 4.2.

- (ii) *M is endowed with an almost CR hermitian structure of codimension r , where D and J are D^\perp -parallel. (In particular, the integral foliation of D^\perp is totally geodesic.)*

Proof. Let us assume (i). For $X \in \Gamma(V^\phi)$,

$$X \cdot \phi = iJ^\phi X \cdot \phi$$

Differentiate with respect to $u \in \Gamma((V^\phi)^\perp)$

$$\nabla_u X \cdot \phi + X \cdot \nabla_u^S \phi = i \nabla_u (J^\phi X) \cdot \phi + i J^\phi X \cdot \nabla_u^S \phi,$$

so that

$$\nabla_u X \cdot \phi = i \nabla_u (J^\phi X) \cdot \phi.$$

Since ϕ is a partially pure spinor

$$\begin{aligned}\nabla_u X &\in V^\phi \\ \nabla_u(J^\phi X) &= J^\phi(\nabla_u X),\end{aligned}$$

i.e. D and J are D^\perp parallel, and so is D^\perp .

Conversely, if M admits a CR hermitian structure, by Theorem 4.1, M admits a twisted Spin structure, a twisted spinor bundle S endowed with a connection ∇^S as described in subsection 4.2, and a partially pure spinor field $\phi \in \Gamma(S)$ such that $V^\phi = D$, $J^\phi = J$ and

$$(X - iJX) \cdot \phi = 0$$

for every $X \in \Gamma(D)$. Let $u \in \Gamma(D^\perp)$ and differentiate

$$X \cdot \phi = iJX \cdot \phi$$

so that

$$\nabla_u X \cdot \phi + X \cdot \nabla_u^S \phi = i\nabla_u(JX) \cdot \phi + iJX \cdot \nabla_u^S \phi.$$

Since J is D^\perp -parallel

$$\nabla_u(JX) = J(\nabla_u X),$$

and

$$X \cdot \nabla_u^S \phi = iJX \cdot \nabla_u^S \phi.$$

i.e.

$$(X - iJX) \cdot \nabla_u^S \phi = 0.$$

□

Example. The almost CR distribution D_7 on $\mathcal{G}_{m,1,r}$ gives the following example for Theorem 4.5

$$\begin{array}{ccc} S^r & \hookrightarrow & \frac{SO(2m+r+1)}{U(m) \times SO(r)} \\ & & \downarrow \\ & & \frac{SO(2m+r+1)}{U(m) \times SO(r+1)}, \end{array}$$

Remark. A generalized D^\perp -Killing partially pure spinor field ϕ is a spinor such that

$$\nabla_u^S \phi = E(u) \cdot \phi,$$

where E is a symmetric endomorphism of D^\perp . Such a spinor also satisfies the hypotheses of Theorem 4.5.

From Theorems 4.3 and 4.5 we obtain the following.

Corollary 4.1. *Let M be an oriented n -dimensional Riemannian manifold. The following are equivalent:*

- (i) *M is locally the Riemannian product of a Kähler manifold and a Riemannian manifold.*
- (ii) *M admits a twisted $\text{Spin}^{c,r}(n)$ structure and a twisted spinor bundle S carrying a partially pure spinor field $\phi \in \Gamma(S)$ satisfying*

$$(Y - iJ^\phi(Y)) \cdot \nabla_Z^S \phi = 0$$

for every $Y \in \Gamma(V^\phi)$ and $Z \in \Gamma(TM)$, where ∇^S is the covariant derivative described in subsection 4.2.

□

In the case of a real D^\perp -Killing partially pure spinor, we can say a little more about the foliation leaves' curvature.

Theorem 4.6. *Let M be a $\text{Spin}^{c,r}$ n -dimensional Riemannian manifold such that its twisted spinor bundle S admits a partially pure spinor field $\phi \in \Gamma(S)$ satisfying*

$$\nabla_u^S \phi = \mu u \cdot \phi$$

for every $u \in \Gamma((V^\phi)^\perp)$, where ∇^S is the connection described in subsection 4.2 and $\mu \in \mathbb{R}$, i.e. ϕ is real Killing in the directions of $(V^\phi)^\perp$. Then

- *The Ricci tensor decomposes as follows*

$$\text{Ric}^{(V^\phi)^\perp} = 4(r-1)\mu^2 \text{Id}_{(V^\phi)^\perp} + \sum_{1 \leq k < l \leq r} \left[\text{proj}_{(V^\phi)^\perp} \circ \hat{\Theta}_{kl}|_{(V^\phi)^\perp} \right] \circ \hat{\eta}_{kl}^\phi,$$

where Θ_{kl} denote the local curvature 2-forms corresponding to the auxiliary connection on the $SO(r)$ principal bundle.

- *The scalar curvature of each leaf tangent to $(V^\phi)^\perp$ is given by*

$$R^{(V^\phi)^\perp} = 4r(r-1)\mu^2 + \sum_{1 \leq k < l \leq r} \text{tr} \left(\left[\text{proj}_{(V^\phi)^\perp} \circ \hat{\Theta}_{kl}|_{(V^\phi)^\perp} \right] \circ \hat{\eta}_{kl}^\phi \right).$$

- *If*

$$\sum_{1 \leq k < l \leq r} \left[\text{proj}_{(V^\phi)^\perp} \circ \hat{\Theta}_{kl}|_{(V^\phi)^\perp} \right] \circ \hat{\eta}_{kl}^\phi = \lambda \text{Id}_{(V^\phi)^\perp}$$

along a leaf of the foliation tangent to $(V^\phi)^\perp$ for some constant $\lambda \in \mathbb{R}$, then the leaf is Einstein.

Proof. Since ϕ is partially pure, $n = 2m + r$ where $\text{rank}(V^\phi) = 2m$ and $\text{rank}((V^\phi)^\perp) = r$. Let (e_1, \dots, e_n) and (f_1, \dots, f_r) be local orthonormal frames of TM and F respectively, such that

$$\begin{aligned} V^\phi &= \text{span}(e_1, \dots, e_{2m}), \\ e_{2j} &= J^\phi(e_{2j-1}), \\ (V^\phi)^\perp &= \text{span}(e_{2m+1}, \dots, e_{2m+r}), \\ \eta_{kl}^\phi &= e_{2m+k} \wedge e_{2m+l}, \end{aligned}$$

for $1 \leq j \leq m$ and $1 \leq k < l \leq r$. First, if $u, v \in \Gamma((V^\phi)^\perp)$,

$$R^{\theta,A}(u, v)\phi = \mu^2(v \cdot u - u \cdot v) \cdot \phi$$

Now, for $2m+1 \leq i, j \leq 2m+r$,

$$\sum_{i=2m+1}^{2m+r} e_i \cdot R^{\theta,A}(e_j, e_i)(\phi) = -2(r-1)\mu^2 e_j \cdot \phi.$$

By taking the real part of the hermitian product with $e_t \cdot \phi$ we get

$$\text{Re} [-2(r-1)\mu^2 \langle e_j \cdot \phi, e_t \cdot \phi \rangle] = -2(r-1)\mu^2 \delta_{jt}.$$

If $u, v \in \Gamma((V^\phi)^\perp)$ then, by Theorem 4.5, $[u, v] \in \Gamma((V^\phi)^\perp)$, and

$$\begin{aligned} R^M(u, v)e_i &\in \Gamma(V^\phi) && \text{if } 1 \leq i \leq 2m, \\ R^M(u, v)e_i &\in \Gamma((V^\phi)^\perp) && \text{if } 2m+1 \leq i \leq 2m+r. \end{aligned}$$

so that

$$\begin{aligned}\langle R^M(u, v)e_i, e_j \rangle &= 0 & \text{if } 1 \leq i \leq 2m, 2m+1 \leq j \leq 2m+r, \\ \langle R^M(u, v)e_i, e_j \rangle &= 0 & \text{if } 2m+1 \leq i \leq 2m+r, 1 \leq j \leq 2m.\end{aligned}$$

Now, if $1 \leq \alpha \leq r$,

$$\begin{aligned}R^{\theta, A}(u, e_{2m+\alpha})\phi &= \frac{1}{2} \sum_{1 \leq i < j \leq n} \langle R^M(u, e_{2m+\alpha})e_i, e_j \rangle e_i e_j \cdot \phi \\ &\quad + \frac{1}{2} \sum_{1 \leq k < l \leq r} \Theta_{kl}(u, e_{2m+\alpha}) \kappa_{r*}(f_k f_l) \cdot \phi + \frac{i}{2} dA(u, e_{2m+\alpha})\phi \\ &= \frac{1}{2} \sum_{1 \leq i < j \leq 2m} \langle R^M(u, e_{2m+\alpha})e_i, e_j \rangle e_i e_j \cdot \phi \\ &\quad + \frac{1}{2} \sum_{1 \leq k < l \leq r} \langle R^M(u, e_{2m+\alpha})e_{2m+k}, e_{2m+l} \rangle e_{2m+k} e_{2m+l} \cdot \phi \\ &\quad + \frac{1}{2} \sum_{1 \leq k < l \leq r} \Theta_{kl}(u, e_{2m+\alpha}) \kappa_{r*}(f_k f_l) \cdot \phi + \frac{i}{2} dA(u, e_{2m+\alpha})\phi,\end{aligned}$$

where dA denotes the curvature 2-form of the auxiliary connection on the $U(1)$ -principal bundle. Multiply by $e_{2m+\alpha}$ and sum over α , $1 \leq \alpha \leq r$,

$$\begin{aligned}\sum_{\alpha=1}^r e_{2m+\alpha} \cdot R^{\theta, A}(u, e_{2m+\alpha})\phi &= \sum_{\alpha=1}^r \sum_{1 \leq i < j \leq 2m} \langle R^M(u, e_{2m+\alpha})e_i, e_j \rangle e_{2m+\alpha} \cdot e_i e_j \cdot \phi \\ &\quad + \sum_{\alpha=1}^r \sum_{1 \leq k < l \leq r} \langle R^M(u, e_{2m+\alpha})e_{2m+k}, e_{2m+l} \rangle e_{2m+\alpha} \cdot e_{2m+k} e_{2m+l} \cdot \phi \\ &\quad + \sum_{\alpha=1}^r \sum_{1 \leq k < l \leq r} \Theta_{kl}(u, e_{2m+\alpha}) e_{2m+\alpha} \cdot \kappa_{r*}(f_k f_l) \cdot \phi \\ &\quad + i \sum_{\alpha=1}^r dA(u, e_{2m+\alpha}) e_{2m+\alpha} \cdot \phi.\end{aligned}$$

Furthermore,

$$\begin{aligned}&\text{Re} \left\langle \sum_{\alpha=1}^r e_{2m+\alpha} \cdot R^{\theta, A}(e_{2m+\gamma}, e_{2m+\alpha})(\phi), e_{2m+\beta} \cdot \phi \right\rangle \\ &= \text{Re} \left\langle \sum_{\alpha=1}^r \sum_{1 \leq i < j \leq 2m} \langle R^M(e_{2m+\gamma}, e_{2m+\alpha})e_i, e_j \rangle e_{2m+\alpha} \cdot e_i e_j \cdot \phi, e_{2m+\beta} \cdot \phi \right\rangle \\ &\quad + \text{Re} \left\langle \sum_{\alpha=1}^r \sum_{1 \leq k < l \leq r} \langle R^M(e_{2m+\gamma}, e_{2m+\alpha})e_{2m+k}, e_{2m+l} \rangle e_{2m+\alpha} \cdot e_{2m+k} e_{2m+l} \cdot \phi, e_{2m+\beta} \cdot \phi \right\rangle \\ &\quad + \text{Re} \left\langle \sum_{\alpha=1}^r \sum_{1 \leq k < l \leq r} \Theta_{kl}(e_{2m+\gamma}, e_{2m+\alpha}) e_{2m+\alpha} \cdot \kappa_{r*}(f_k f_l) \cdot \phi, e_{2m+\beta} \cdot \phi \right\rangle \\ &\quad + \text{Re} \left\langle i \sum_{\alpha=1}^r dA(e_{2m+\gamma}, e_{2m+\alpha}) e_{2m+\alpha} \cdot \phi, e_{2m+\beta} \cdot \phi \right\rangle \\ &= - \left\langle \text{Ric}^{(V^\phi)^\perp}(e_{2m+\alpha}), e_{2m+\beta} \right\rangle |\phi|^2\end{aligned}$$

$$\begin{aligned}
& + \sum_{\alpha=1}^r \sum_{1 \leq k < l \leq r} \Theta_{kl}(e_{2m+\gamma}, e_{2m+\alpha}) \text{Re} \langle e_{2m+\alpha} \cdot \kappa_{r*}(f_k f_l) \cdot \phi, e_{2m+\beta} \cdot \phi \rangle \\
& = -\text{Ric}_{2m+\beta, 2m+\gamma}^{(V^\phi)^\perp} + \sum_{\alpha=1}^r \sum_{1 \leq k < l \leq r} \Theta_{kl}(e_{2m+\gamma}, e_{2m+\alpha}) \eta_{kl}^\phi(e_{2m+\alpha}, e_{2m+\beta}),
\end{aligned}$$

i.e.

$$\text{Ric}^{(V^\phi)^\perp} = 4(r-1)\mu^2 \text{Id}_{(V^\phi)^\perp} + \sum_{1 \leq k < l \leq r} \left[\text{proj}_{(V^\phi)^\perp} \circ \hat{\Theta}_{kl}|_{(V^\phi)^\perp} \right] \circ \hat{\eta}_{kl}^\phi.$$

□

4.6 CR foliation with Equidistant leaves

Theorem 4.7. *Let M be a $\text{Spin}^{c,r}$ Riemannian manifold such that its twisted spinor bundle S admits a partially pure spinor field $\phi \in \Gamma(S)$. Let V^ϕ denote the almost-CR distribution and $(V^\phi)^\perp$ its orthogonal distribution. If*

$$\begin{aligned}
(X - iJ^\phi X) \cdot \nabla_u^S \psi &= 0, \\
(X - iJ^\phi X) \cdot \nabla_X^S \psi &= 0,
\end{aligned}$$

for all $X \in \Gamma(V^\phi)$ and $u \in \Gamma((V^\phi)^\perp)$, where ∇^S the covariant derivative described in subsection 4.2, then

- V^ϕ , J^ϕ and $(V^\phi)^\perp$ are $(V^\phi)^\perp$ -parallel;
- the totally geodesic foliation tangent to $(V^\phi)^\perp$ has equidistant leaves.

Furthermore, if the complex structure J descends to the space of leaves N at regular points, such a complex structure is nearly-Kähler structure.

Proof. The first statement follows from Theorem 4.5. Recall the condition for a foliation to have equidistant leaves [7, Proposition 7]

$$\langle \nabla_X u, Y \rangle + \langle X, \nabla_Y u \rangle = 0, \tag{13}$$

for every $X, Y \in \Gamma(D)$ and $u \in \Gamma(D^\perp)$. Since $\langle u, Y \rangle = \langle u, X \rangle = 0$,

$$\langle \nabla_X u, Y \rangle + \langle u, \nabla_X Y \rangle = 0,$$

$$\langle \nabla_Y u, X \rangle + \langle u, \nabla_Y X \rangle = 0,$$

so that (13) becomes

$$\langle \nabla_X Y + \nabla_Y X, u \rangle = 0.$$

We must prove $\nabla_X Y + \nabla_Y X \in \Gamma(D)$. Taking covariant derivative with respect to Y on

$$X \cdot \psi = iJ(X) \cdot \psi,$$

and with respect to X on

$$Y \cdot \psi = iJ(Y) \cdot \psi.$$

we get

$$\nabla_Y X \cdot \psi + X \cdot \nabla_Y^S \psi = i\nabla_Y(J(X)) \cdot \psi + iJ(X) \cdot \nabla_Y^S \psi,$$

$$\nabla_X Y \cdot \psi + Y \cdot \nabla_X^S \psi = i\nabla_X(J(Y)) \cdot \psi + iJ(Y) \cdot \nabla_X^S \psi.$$

Rearranging terms

$$\begin{aligned} \nabla_Y X \cdot \psi + (X - iJ(X)) \cdot \nabla_Y^S \psi &= i\nabla_Y(J(X)) \cdot \psi, \\ \nabla_X Y \cdot \psi + (Y - iJ(Y)) \cdot \nabla_X^S \psi &= i\nabla_X(J(Y)) \cdot \psi. \end{aligned}$$

Adding up the last two equations and using

$$\begin{aligned} 0 &= ((X + Y) - iJ(X + Y)) \cdot \nabla_{X+Y}^S \psi \\ &= (X - iJ(X)) \cdot \nabla_X \psi + (Y - iJ(Y)) \cdot \nabla_Y^S \psi \\ &\quad + (X - iJ(X)) \cdot \nabla_Y \psi + (Y - iJ(Y)) \cdot \nabla_X^S \psi \\ &= (X - iJ(X)) \cdot \nabla_Y \psi + (Y - iJ(Y)) \cdot \nabla_X^S \psi, \end{aligned}$$

we get

$$(\nabla_Y X + \nabla_X Y) \cdot \psi = i(\nabla_Y(J(X)) + \nabla_X(J(Y))) \cdot \psi.$$

This means that $\nabla_Y X + \nabla_X Y \in \Gamma(D)$, i.e. the foliation has equidistant leaves. Furthermore, we have

$$\nabla_Y(J(X)) + \nabla_X(J(Y)) = J(\nabla_Y X + \nabla_X Y).$$

By setting $X = Y$,

$$\nabla_X(J(X)) = J(\nabla_X X),$$

i.e.

$$(\nabla_X J)(X) = 0.$$

Since the leaves of the foliation are equidistant, by [7, Lemma 19] the leaf space N inherits a Riemannian metric at regular points and the quotient map π is a Riemannian submersion. The Levi-Civita connection at a regular point of N is given by

$$\nabla_x^* y = \pi_*(\nabla_X Y)$$

where $\pi_*(X) = x$ and $\pi_*(Y) = y$.

Finally, if we also have that J descends to a complex structure \mathcal{J} on N in the form $\mathcal{J}(\pi_*(X)) = \pi_*(JX)$ and

$$\begin{aligned} \nabla_x^*(\mathcal{J}(x)) - \mathcal{J}(\nabla_x^* x) &= \pi_* \nabla_X(J(X)) - \mathcal{J}(\pi_*(\nabla_X X)) \\ &= \pi_* \nabla_X(J(X)) - \pi_*(J(\nabla_X X)) \\ &= \pi_*(\nabla_X(J(X)) - J(\nabla_X X)) \\ &= \pi_*(0) \\ &= 0. \end{aligned}$$

i.e. \mathcal{J} is a nearly-Kähler structure at regular points of N . □

5 Codimension 1 almost CR structures

Now we will focus our attention on oriented Riemannian manifolds of dimension $n = 2m + 1$, i.e. $r = 1$. Recall that

$$Spin^{c,1}(n) = Spin^c(n),$$

so we are back to $Spin^c$ geometry.

Let M be a $2m + 1$ -dimensional Spin^c manifold carrying a partially pure spinor field ψ . We have

$$TM = V^\psi \oplus (V^\psi)^\perp,$$

where $(V^\psi)^\perp$ is a trivial real line bundle over M . Consider the vector field ξ^ψ defined by

$$\langle X, \xi^\psi \rangle = i \langle X \cdot \psi, \psi \rangle,$$

for all $X \in \Gamma(TM)$. If $X \in \Gamma(V^\psi)$,

$$X \cdot \psi = iJ^\psi(X) \cdot \psi,$$

so that taking the scalar product with ψ gives

$$\langle X \cdot \psi, \psi \rangle = i \langle J^\psi(X) \cdot \psi, \psi \rangle = 0,$$

i.e.

$$\langle X, \xi^\psi \rangle = 0. \quad (14)$$

Let $\theta^\psi = (\xi^\psi)^*$ the metric dual 1-form, and G_ψ the symmetric 2-form defined by

$$G_\psi(X, Y) = d\theta^\psi(X, J^\psi(Y)),$$

for any $X, Y \in \Gamma(V^\psi)$.

5.1 Pseudoconvex CR manifolds

Theorem 5.1. *Let M be an oriented $2m + 1$ -dimensional smooth manifold. Then, M is a pseudoconvex CR manifold if and only if it has a Spin^c structure carrying an integrable partially pure spinor such that G_ψ is positive definite.*

Proof. Assume that M is a pseudoconvex CR manifold. Then $D = \ker \theta$ for some hermitian structure θ . We consider the Tanaka-Webster (Riemannian) metric g_θ . It is known that (M, g_θ) has a canonical Spin^c structure carrying an integrable partially pure spinor field ψ of unit length for which $D = V^\psi$. Moreover, we recall that there exists a unique vector field T such that $\theta(T) = 1$ and $T \lrcorner d\theta = 0$. We claim that $T = \xi^\psi$ so that $\theta = \theta^\psi$ and G_ψ is positive definite.

Let us note that $\xi^\psi \neq 0$, since

$$T \cdot \psi = -i\psi$$

implies

$$\begin{aligned} g_\theta(T, \xi^\psi) &= i \langle T \cdot \psi, \psi \rangle \\ &= 1. \end{aligned} \quad (15)$$

Since $g_\theta(X, \xi^\psi) = 0$ for all $X \in \Gamma(D)$, ξ^ψ is a multiple of T and by (15)

$$T = \xi^\psi.$$

Conversely, assume that a $2m + 1$ dimensional Riemannian manifold M carries an integrable partially pure spinor field and such that G_ψ is positive definite. It remains to prove that it is a pseudoconvex structure. For this, it is sufficient to prove that $\ker \theta^\psi = V^\psi$ which is already the case by (14). \square

Remark. From the proof of Theorem 5.1, we see that the Tanaka Webster metric is given by

$$g_{\theta^\psi}(X, Y) = G_\psi(X, Y)$$

for $X, Y \in V^\psi$.

5.2 Metric contact and Sasakian manifolds

Let M^{2m+1} be an oriented smooth manifold and $(\mathfrak{X}, \xi, \eta)$ a synthetic object consisting of a $(1,1)$ -tensor field $\mathfrak{X} : TM \rightarrow TM$, a tangent vector field ξ , and a differential 1-form η on M . $(\mathfrak{X}, \xi, \eta)$ is an *almost contact structure* if

$$\mathfrak{X}^2 = -\text{Id} + \eta \otimes \xi, \quad \mathfrak{X}\xi = 0, \quad \eta(\xi) = 1, \quad \eta \circ \mathfrak{X} = 0.$$

An almost contact structure is said to be *normal* if, for all $X, Y \in \Gamma(TM)$, $N^{\mathfrak{X}} = d\eta \otimes \xi$, where $N^{\mathfrak{X}}$ is the Nijenhuis contact tensor of \mathfrak{X} defined by

$$N^{\mathfrak{X}}(X, Y) = -\mathfrak{X}^2[X, Y] + \mathfrak{X}[\mathfrak{X}X, Y] + \mathfrak{X}[X, \mathfrak{X}Y] - [\mathfrak{X}X, \mathfrak{X}Y],$$

for all $X, Y \in \Gamma(TM)$.

A Riemannian metric g is said to be compatible with the almost contact structure if

$$g(\mathfrak{X}X, \mathfrak{X}Y) = g(X, Y) - \eta(X)\eta(Y),$$

for all $X, Y \in \Gamma(TM)$. An almost contact structure $(\mathfrak{X}, \xi, \eta)$ together with a compatible Riemannian metric g is called an *almost contact metric structure*.

Given an almost contact metric structure $(\mathfrak{X}, \xi, \eta, g)$ one defines a 2-form α by $\alpha(X, Y) = g(X, \mathfrak{X}Y)$ for all $X, Y \in \Gamma(TM)$. Now, $(\mathfrak{X}, \xi, \eta, g)$ is said to satisfy the *contact condition* if $\alpha = d\eta$ and if it is the case, $(\mathfrak{X}, \xi, \eta, g)$ is called a *contact metric structure* on M .

A contact metric structure $(\mathfrak{X}, \xi, \eta, g)$ that is also normal is called a *Sasakian structure* (and M is *Sasakian*).

Every strictly pseudoconvex CR manifold is also a contact metric manifold [8, 1]. Moreover, this contact metric structure is a Sasakian structure if and only if the Tanaka torsion vanishes, i.e., $\tau = 0$ [8, 1]. Conversely, a metric contact manifold has a natural almost CR structure [8, 1]. This almost CR structure is a CR structure (and then automatically strictly pseudoconvex) if and only if $\mathfrak{X} \circ N^{\mathfrak{X}}(X, Y) = 0$ for all $X, Y \in \Gamma(\ker \eta)$ [17].

Corollary 5.1. *Every oriented contact Riemannian manifold has a Spin^c structure carrying a partially pure spinor. Moreover, this spinor is integrable if the contact structure is normal, i.e., if M is Sasakian.*

Proof. We know that any contact Riemannian manifold has a Spin^c structure. Moreover, for this Spin^c structure carries a partially pure spinor. We have

$$(N^{\mathfrak{X}}(X, Y) + \eta([X, Y])\xi) \cdot \psi = (X - i\mathfrak{X}X)\nabla_{Y - i\mathfrak{X}Y}\psi - (Y - i\mathfrak{X}Y)\nabla_{X - i\mathfrak{X}X}\psi,$$

for all $X, Y \in \Gamma(V^\psi)$. But $d\eta(X, Y) = -\eta([X, Y])$ for all $X, Y \in \Gamma(\ker \eta)$. Hence, if the contact metric is normal, ψ is integrable. \square

Corollary 5.2. *Let M be an oriented contact Riemannian manifold. Then, it is a Spin^c manifold carrying an integrable partially pure spinor ψ with positive definite G_ψ if and only if $\mathfrak{X} \circ N^{\mathfrak{X}} = 0$. \square*

Corollary 5.3. *Let M be a Sasakian manifold satisfying $N^{\mathfrak{X}} \circ \mathfrak{X} = 0$. Then, it is a Spin^c manifold carrying a integrable partially pure spinor field with positive definite G_ψ . Conversely, if M is a Riemannian Spin^c manifold carrying a integrable strictly partially pure spinor field with positive definite G_ψ and such that $\tau = 0$, then M is a Sasakian manifold. \square*

5.3 Isometric immersions via partially pure Spin^c spinors

Let N^{2m-1} be an oriented real hypersurface of a Kähler manifold (M^{2m}, \bar{g}, J) endowed with the metric g induced by \bar{g} . We denote by ν the unit normal inner vector globally defined on M and by Π the second fundamental form of the immersion.

The complex structure J induces an almost contact metric structure $(\mathfrak{X}, \xi, \eta, g)$ on M , where \mathfrak{X} is the $(1, 1)$ -tensor defined by

$$g(\mathfrak{X}X, Y) = \bar{g}(JX, Y)$$

for all $X, Y \in \Gamma(TN)$, $\xi = -J\nu$ is a tangent vector field and η is the 1-form associated with ξ , that is

$$\eta(X) = g(\xi, X).$$

For every $X \in \Gamma(TN)$

$$\mathfrak{X}^2 X = -X + \eta(X)\xi, \quad g(\xi, \xi) = 1, \quad \text{and} \quad \mathfrak{X}\xi = 0.$$

Moreover, from the relation between the Riemannian connections $\bar{\nabla}$ of M and ∇ of N ,

$$\bar{\nabla}_X Y = \nabla_X Y + g(\Pi(X), Y)\nu,$$

we deduce the following two identities:

$$\begin{aligned} (\nabla_X \mathfrak{X})Y &= \eta(Y)\Pi(X) - g(\Pi(X), Y)\xi \\ \nabla_X \xi &= \mathfrak{X}(\Pi(X)), \end{aligned}$$

for every $X, Y \in \Gamma(TN)$. We can choose an orthonormal frame

$$\mathcal{B} = \{e_1, e_2 = \mathfrak{X}e_1, \dots, e_{2m-3}, e_{2m-2} = \mathfrak{X}e_{2m-3}, \xi\}$$

of N such that

$$\mathcal{B} \cup \{\nu = J\xi\}$$

is an orthonormal frame of M .

Theorem 5.2. *Let (M^{2m}, g, J) be a Hermitian manifold. Then, any real oriented hypersurface $N \subset M$ has a Spin^c structure carrying an integrable partially pure spinor.*

Proof. Since M is a Hermitian manifold, it has a canonical Spin^c structure carrying a integrable pure spinor field ψ . The restriction of this Spin^c structure to an oriented real hypersurface N^{2m-1} gives a Spin^c structure carrying a spinor field $\phi = \psi|_N$ satisfying

$$\nabla_X^N \phi = \nabla_X^M \psi|_M - \frac{1}{2}\Pi(X) \bullet \phi$$

for all $X \in \Gamma(TN)$. We claim that the spinor field ϕ is an integrable partially pure spinor field. For $j = 1, \dots, 2m-2$, we have

$$\begin{aligned} (e_j - i\mathfrak{X}e_j) \bullet \phi &= (e_j - i\mathfrak{X}e_j) \cdot \nu \cdot \psi|_M \\ &= -\nu \cdot (e_j - i\mathfrak{X}e_j) \cdot \psi|_M \\ &= 0, \end{aligned}$$

since

$$(e_j - i\mathfrak{X}e_j) \cdot \psi = (e_j - iJe_j) \cdot \psi$$

$$= 0.$$

The distribution

$$V^\phi = \{X \in \Gamma(TN), \quad X \bullet \phi = -i\mathfrak{X}X \bullet \phi\},$$

is of constant rank $2(m-1)$ and is an almost CR structure. By the Spin^c Gauss formula, the spinor ϕ is integrable if and only if ψ is integrable because

$$(X - iJX) \bullet \Pi(Y - iJY) \bullet \phi = (Y - iJY) \bullet \Pi(X - iJX) \bullet \phi,$$

for all $X, Y \in \Gamma(V^\phi)$. Since ψ is integrable, ϕ is also integrable. \square

Remark. From Proposition 5.2, any real oriented hypersurface N of a Kähler manifold M has a Spin^c structure carrying an integrable partially pure spinor ϕ satisfying

$$\nabla_X^N \phi = -\frac{1}{2} \Pi(X) \bullet \phi,$$

for all $X \in \Gamma(TN)$, i.e. a generalized Killing spinor.

Theorem 5.3. *Let (N^{2m+1}, g) be an oriented almost contact metric Spin^c manifold carrying a parallel partially pure spinor ψ , and $I = [0, 1]$. Then the product $\mathcal{Z} := M \times I$ endowed with the metric $dt^2 + g$ and the Spin^c structure arising from the given one on M is a Kähler manifold having a parallel spinor ψ whose restriction to M is just ϕ .*

Proof. First, the pull back of the Spin^c structure on M defines a Spin^c structure on $M \times I$. Moreover, from the spinor field ϕ , we can construct on $M \times I$ a parallel spinor ψ . It remains to show that $M \times I$ is Kähler. We define the endomorphism \overline{J} by

$$\begin{aligned} \overline{J}(X) &= J(X) \quad \text{for any } X \in \Gamma(D), \\ \overline{J}(T) &= \nu, \\ \overline{J}(\nu) &= -T. \end{aligned}$$

It is easy to prove that $(M \times I, \overline{J}, g + dt^2)$ is an almost Hermitian manifold. Moreover, since $T \bullet \phi = -i\phi$, then

$$\nabla_T T = \nabla_X T = \nabla_{JX} T = 0.$$

Since the immersion is totally geodesic, we get

$$\overline{\nabla}_T T = \overline{\nabla}_X T = \overline{\nabla}_{JX} T = 0. \tag{16}$$

Now, since ϕ is parallel on M , we get for any $X \in \Gamma(D)$,

$$\begin{aligned} \nabla_X J &= 0, \\ \nabla_T X &\in \Gamma(D), \\ \nabla_T JX &= J(\nabla_T X). \end{aligned} \tag{17}$$

Finally, using (16) and (17), we conclude that $\overline{\nabla} \overline{J} = 0$ on $M \times I$. \square

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